Jacobs University | March 23, 2018
Spring 2018

Foundations of Mathematical Physics

Midterm Exam

Instructions:

- Do all the work on this exam paper.
- Show your work, i.e., carefully write down the steps of your solution.
- Calculators and other electronic devices and notes are not allowed.
- You are free to refer to any results proven in class or the homework sheets unless otherwise stated (and unless the problem is to reproduce a result from class or the homework sheets).

Name: ________________________________

Problem 1: Dilations [20 points]

Let $\sigma > 0$. We define the L^2 dilation with σ by $D_\sigma : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$, $f(x) \mapsto (D_\sigma f)(x) = \sigma^{-d/2} f\left(\frac{x}{\sigma}\right)$, where $\mathcal{S}(\mathbb{R}^d)$ is the space of Schwartz functions.

(a) Prove that $D_\sigma : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ is continuous.

(b) Show that $\|D_\sigma f\|_{L^2(\mathbb{R}^d)} = \|f\|_{L^2(\mathbb{R}^d)}$ for all $f \in \mathcal{S}(\mathbb{R}^d)$.

(c) Compute $\mathcal{F} D_\sigma f$ for $f \in \mathcal{S}(\mathbb{R}^d)$, where $\mathcal{F}$ denotes Fourier transform.

Problem 1: Extra Space

Problem 2: The Free Schrödinger Equation [30 points]

We consider the solution to the free Schrödinger equation

$$\psi(t,x) = (2\pi i t)^{-d/2} \int_{\mathbb{R}^d} e^{i\frac{(x-y)^2}{2t}} \psi_0(y)\, dy$$

for ψ_0 in some function space. In the following, $\mathcal{F}$ denotes Fourier transform.

(a) Let $\psi_0 \in \mathcal{S}(\mathbb{R}^d)$ (the space of Schwartz functions). Prove that we can decompose

$$\psi(t,x) = (it)^{-d/2} e^{i\frac{x^2}{2t}} (\mathcal{F}\psi_0)\left(\frac{x}{t}\right) + r(t,x),$$

with $\lim_{t\to\infty} \|r(t,\cdot)\|_{L^2(\mathbb{R}^d)} = 0$.

(b) The free propagator

$$P_f(t) : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d), P_f(t) = \mathcal{F}^{-1} e^{-i\frac{k^2}{2}t} \mathcal{F}$$

is well-defined by extending the Fourier transform from $\mathcal{S}(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$. Is P_f continuous in norm (i.e., uniformly)? Is it strongly (i.e., pointwise) continuous? Is it weakly continuous? Under which conditions (if at all) is P_f differentiable (in norm, strongly, weakly)? Prove all your answers!

Problem 2: Extra Space

Problem 3: Linear Operators [20 points]

(a) Let X and Y be Banach spaces and let $L : X \to Y$ be linear. Prove that L is continuous if and only if L is bounded.

(b) Let $(\varphi_n)_{n\in\mathbb{N}}$ be an orthonormal basis of a Hilbert space $\mathcal{H}$. We define a sequence $(A_n)_{n\in\mathbb{N}}$ of bounded linear operators in $\mathcal{H}$ by

$$A_n\psi = \sum_{i=1}^{\infty} \langle \psi, \varphi_i \rangle \, \varphi_{i+n}$$

for all $\psi \in \mathcal{H}$, where $\langle \cdot, \cdot \rangle$ is the scalar product on $\mathcal{H}$. Prove that A_n converges weakly to 0, but not strongly.

Problem 3: Extra Space