

Foundations of Mathematical Physics

Homework 3

Due on October 6, 2021

Problem 1 [6 points]: Heat Equation

(a) Let $\psi_0 \in \mathcal{S}(\mathbb{R}^d)$. Determine the solution to the heat equation

$$\begin{aligned}\partial_t \psi(t, x) &= \Delta_x \psi(t, x) && \text{for all } (t, x) \in [0, \infty) \times \mathbb{R}^d, \\ \psi(0, x) &= \psi_0(x) && \text{for all } x \in \mathbb{R}^d\end{aligned}$$

by using the Fourier transform. Write the solution as

$$\psi(t, x) = \int_{\mathbb{R}^d} K(t, x - y) \psi_0(y) dy, \quad (1)$$

and explicitly state what the function $K : (0, \infty) \times \mathbb{R}^d \rightarrow \mathbb{R}$ is.

(b) Let $\psi_0 \in C(\mathbb{R}^d)$ be bounded. Show that Equation (1) defines a bounded function $\psi \in C^\infty((0, \infty) \times \mathbb{R}^d)$ which solves the heat equation on $(0, \infty) \times \mathbb{R}^d$. Show also that ψ can be continuously extended by ψ_0 at $t = 0$, i.e., show that $\lim_{t \rightarrow 0} \psi(t, x) = \psi_0(x)$ for all $x \in \mathbb{R}^d$. (*Hint: Use Problem 4 from Homework 2.*)

Problem 2 [6 points]: Multiplication Operators on L^p

Let $V : \mathbb{R}^d \rightarrow \mathbb{R}$ be measurable and $1 \leq p \leq \infty$. Show that V defines a continuous multiplication operator

$$M_V : L^p(\mathbb{R}^d) \rightarrow L^p(\mathbb{R}^d), \psi \mapsto V\psi$$

if and only if $V \in L^\infty(\mathbb{R}^d)$. Show that then

$$\|M_V\|_{\mathcal{L}(L^p)} := \sup_{\|f\|_{L^p}=1} \|M_V f\|_{L^p} = \|V\|_\infty.$$

Problem 3 [8 points]: Convolution in L^p

Let $1 \leq p < \infty$ and $f \in L^p(\mathbb{R}^d)$.

(a) Using the Hölder inequality

$$\|fg\|_{L^1} \leq \|f\|_{L^p} \|g\|_{L^q},$$

where $1/p + 1/q = 1$, show that for $g \in L^1(\mathbb{R}^d)$ we have

$$\|f * g\|_{L^p} \leq \|f\|_{L^p} \|g\|_{L^1}.$$

*Hint: Show first that $f * g \in L^p(\mathbb{R}^d)$ by using $L^p(\mathbb{R}^d) = (L^q(\mathbb{R}^d))'$ (dual space of L^q , $1/p + 1/q = 1$). The inequality can then be shown to follow from this consequence of the Hahn-Banach theorem: For all $h \in L^p(\mathbb{R}^d)$ there is an $\tilde{h} \in L^q(\mathbb{R}^d)$ with $\|\tilde{h}\|_{L^q} = 1$ and*

$$\|h\|_{L^p} = \tilde{h}(h) := \int_{\mathbb{R}^d} \tilde{h}(x)h(x)dx.$$

- (b) Let $\varphi \in C_c^\infty(\mathbb{R}^d)$ with $\varphi \geq 0$ and $\int_{\mathbb{R}^d} \varphi = 1$. Define $f_\sigma := f * (D_\sigma^1 \varphi)$ as in Problem 3 from Homework 2. Using (a), show that f_σ converges to f in $L^p(\mathbb{R}^d)$ as $\sigma \rightarrow 0$, i.e.,

$$\lim_{\sigma \rightarrow 0} \|f_\sigma - f\|_{L^p} = 0.$$

Hint: Use that $C_c(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$ and Problem 4 from Homework 2.