Last time: We started to compute $\mathbb{P}\left(\frac{X(t)}{t}\in\Gamma\right)=\int_{t\Gamma}|\psi(t,x)|^2dx$ in the limit $t\to\infty$, given $\Gamma\subset\mathbb{R}^d$ and for $\psi(t,x)$ the solution to the free SE with initial condition $\psi(0,x)=\psi_0(x)$.

We computed: $\psi(t,x)=\frac{e^{i\frac{x^2}{2t}}}{(it)^{d/2}}\hat\psi_0\left(\frac{x}{t}\right)+r(t,x)$, with $r(t,x)=\frac{e^{i\frac{x^2}{2t}}}{(it)^{d/2}}\hat h_t\left(\frac{x}{t}\right)$,

$$h_t(y):=\left(e^{i\frac{y^2}{2t}}-1\right)\psi_0(y).$$

$|z+w|^2=(z+w)(\bar z+\bar w)=|z|^2+|w|^2+\underbrace{\bar z w+z\bar w}_{=2\mathrm{Re}\,\bar z w}$

$$\Rightarrow \int_{t\Gamma}|\psi(t,x)|^2dx$$

$$=t^{-d}\int_{t\Gamma}\left|\hat\psi_0\left(\tfrac{x}{t}\right)\right|^2dx+\overbrace{\int_{t\Gamma}|r(t,x)|^2dx}^{=t^{-d}\int_{t\Gamma}\left|\hat h_t\left(\frac{x}{t}\right)\right|^2dx}+2\mathrm{Re}\,t^{-d}\int_{t\Gamma}\overline{\hat\psi_0\left(\tfrac{x}{t}\right)}\,\hat h_t\left(\tfrac{x}{t}\right)dx$$

change of variables $\frac{x}{t}=p$

$$=\int_\Gamma|\hat\psi_0(p)|^2dp+\underbrace{\int_\Gamma|\hat h_t(p)|^2dp}+\underbrace{2\mathrm{Re}\int_\Gamma\overline{\hat\psi_0(p)}\,\hat h_t(p)dp}$$

Cauchy-Schwarz: $\leq 2\,\|\hat\psi_0(p)\|_{L^2}\underbrace{\|\hat h_t\|_{L^2}}_{\to 0 \text{ as } t\to\infty}$

$$\int_\Gamma|\hat h_t(p)|^2dp\leq\int_{\mathbb{R}^d}|\hat h_t(y)|^2dy\overset{\text{Plancherel}}{=}\int_{\mathbb{R}^d}|h_t(y)|^2dy=\int\left|e^{i\frac{y^2}{2t}}-1\right|^2|\psi_0(y)|^2dy\xrightarrow{t\to\infty}0$$

by dominated convergence

(integrand $\to 0$ pointwise and bounded by $4|\psi_0|\in L^1$)

We thus have proven:

Theorem 2.42:

Let $\psi(t,x)$ be the solution to the free SE with initial condition $\psi_0\in S$, let $\Gamma\subset\mathbb{R}^d$ be measurable. Then $\lim_{t\to\infty}\mathbb{P}\left(\frac{X(t)}{t}\in\Gamma\right):=\lim_{t\to\infty}\int_{t\Gamma}|\psi(t,x)|^2dx=\int_\Gamma|\hat\psi_0(p)|^2dp.$

Remarks:

- Recall $\psi(t_0,x) = \mathcal{F}^{-1} e^{-i\frac{k^2}{2}t_0} \mathcal{F}\psi_0(x)$, so $\hat{\psi}(t_0,k) = e^{-i\frac{k^2}{2}t_0}\hat{\psi}_0(k)$

  $\Rightarrow |\hat{\psi}(t_0,k)|^2 = |e^{-i\frac{k^2}{2}t_0}\hat{\psi}_0(k)| = |\hat{\psi}_0(k)|$, so Theorem 2.42 is independent of choice of initial time t_0.

  see Session 9

  Also: For $\psi_{0,a}(x) := \psi_0(x-a) = (e^{-ia(-i\nabla)}\psi_0)(x)$, we find

  $|\hat{\psi}_{0,a}(k)| = |\mathcal{F}\psi_{0,a}(k)| = |e^{-iak}\hat{\psi}_0(k)| = |\hat{\psi}_0(k)|$, so Theorem 2.42 is independent of translations.

- Expectation value of asymptotic momentum:

$$\mathbb{E} = \int p\,|\hat{\psi}_0(p)|^2 dp = \int \overline{\hat{\psi}_0(p)}\, p\, \hat{\psi}_0(p)\,dp = \int \overline{\psi(t,x)}\,(-i\nabla)\,\psi(t,x)\,dx$$

$= \langle \psi_t, \hat{P}\psi_t\rangle$ where $\hat{P} = -i\nabla_x$ is called "momentum operator".

Let us next discuss Theorem 2.42 from the point of view of scaling limits.

Consider macroscopic scales $\underbrace{\tilde{x} = x\varepsilon}$ and $\tilde{t} = t\varepsilon$ for small ε ($\varepsilon \to 0$ later)

If ε very small, x needs to be very large ($\tilde{x} = x\varepsilon$ is order 1 for x of order $\frac{1}{\varepsilon}$)

Then $\psi(t,x) = \psi(\frac{\tilde{t}}{\varepsilon}, \frac{\tilde{x}}{\varepsilon}) =: \varepsilon^{\frac{d}{2}}\psi_\varepsilon(\tilde{t},\tilde{x})$, where we have defined the macroscopic wave function

$$\psi_\varepsilon(t,x) = \varepsilon^{-\frac{d}{2}}\underbrace{\psi(\tfrac{t}{\varepsilon}, \tfrac{x}{\varepsilon})}_{\text{for } t,x \sim \text{order } 1,\ \psi \text{ is evaluated at macroscopically large } \frac{t}{\varepsilon}, \frac{x}{\varepsilon}}$$

for convenience, we forget about the tilde in the notation from now on

The $\varepsilon^{-\frac{d}{2}}$ factor is included so that $\|\psi_\varepsilon(t,\cdot)\|_{L^2} = \|\psi(\frac{t}{\varepsilon},\cdot)\|_{L^2} = \|\psi_0\|_{L^2}$.

$$\Rightarrow i\partial_t \psi_\varepsilon(t,x) = \varepsilon^{-\frac{d}{2}}\, i\, \underbrace{\frac{1}{\varepsilon}\frac{\partial}{\partial(\frac{t}{\varepsilon})}\psi(\tfrac{t}{\varepsilon}, \tfrac{x}{\varepsilon})}$$

$$= i\frac{\partial\psi}{\partial\tilde{t}}(\tilde{t},\tilde{x})\Big|_{\substack{\tilde{t}=\frac{t}{\varepsilon}\\ \tilde{x}=\frac{x}{\varepsilon}}} = -\frac{1}{2}\Big(\sum_{i=1}^{d}\frac{\partial^2}{\partial\tilde{x}_i^2}\psi\Big)(\tilde{t},\tilde{x})\Big|_{\substack{\tilde{t}=\frac{t}{\varepsilon}\\ \tilde{x}=\frac{x}{\varepsilon}}} = -\frac{1}{2}\Big(\sum_{i=1}^{d}\frac{\partial^2}{\partial(\frac{x}{\varepsilon})^2}\Big)\psi(\tfrac{t}{\varepsilon}, \tfrac{x}{\varepsilon})$$

$$= -\frac{1}{2}\varepsilon^2\sum_{i=1}^{d}\frac{\partial^2}{\partial x_i^2}\psi(\tfrac{t}{\varepsilon}, \tfrac{x}{\varepsilon})$$

$$\Rightarrow i\partial_t \Psi_\varepsilon(t,x) = \varepsilon\left(-\tfrac{1}{2}\Delta_x\right)\Psi_\varepsilon(t,x) \qquad \Rightarrow i\varepsilon\partial_t \Psi_\varepsilon(t,x) = -\frac{\varepsilon^2}{2}\Delta_x \Psi_\varepsilon(t,x)$$

Recall that in SI units the SE is $i\hbar\partial_t \Psi(t,x) = -\frac{\hbar^2}{2m}\Delta_x \Psi(t,x)$, so formally

$\lim_{\varepsilon\to 0}$ is the same as $\lim_{\hbar\to 0}$ (which people often consider although $\hbar$ is a physical constant)

Our previous computation then reads: $\Psi_\varepsilon(t,x) = \dfrac{e^{i\frac{x^2}{2\varepsilon t}}}{(it)^{d/2}}\hat{\Psi}_0\left(\frac{x}{t}\right) + \underbrace{\varepsilon^{-\frac{d}{2}} r\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon}\right)}_{=:r_\varepsilon(t,x)}$

with $\|r_\varepsilon\|^2 = \varepsilon^{-d}\int |r(\frac{t}{\varepsilon},\frac{x}{\varepsilon})|^2 dx$

$= \int |r(\frac{t}{\varepsilon},y)|^2 dy \xrightarrow{\varepsilon\to 0} 0$

i.e., $|\hat{\Psi}_0(p)|^2$ is the asymptotic momentum distribution as $\varepsilon\to 0$.

Next, we establish a more direct connection to velocity:

Consider the probability density $\rho_\Psi(t,x) = |\Psi(t,x)|^2$.

$$\Rightarrow \partial_t \rho_\Psi(t,x) = \partial_t |\Psi(t,x)|^2 = \overline{(\partial_t \Psi(t,x))}\,\Psi(t,x) + \overline{\Psi(t,x)}\,(\partial_t \Psi(t,x))$$

$$= \frac{i}{2}\overline{(-\Delta\Psi(t,x))}\,\Psi(t,x) - \frac{i}{2}\overline{\Psi(t,x)}\,(-\Delta\Psi(t,x))$$

$\frac{i}{2}\bar{z} - \frac{i}{2}z = \operatorname{Im} z$

$$= \operatorname{Im} \overline{\Psi(t,x)}\,(-\Delta\Psi(t,x))$$

$$= -\nabla \underbrace{\operatorname{Im} \overline{\Psi(t,x)}\,(\nabla\Psi(t,x))}_{=:j_\Psi(t,x) = \text{current}} \qquad (\text{since } \overline{\nabla\Psi}\,\nabla\Psi \in \mathbb{R})$$

$\Rightarrow \boxed{\partial_t \rho_\Psi + \nabla j_\Psi = 0 \text{, continuity equation}}$ (Note: also holds if $i\partial_t\Psi = -\frac{\Delta}{2}\Psi + V\Psi$ with $V(x)\in\mathbb{R}$)