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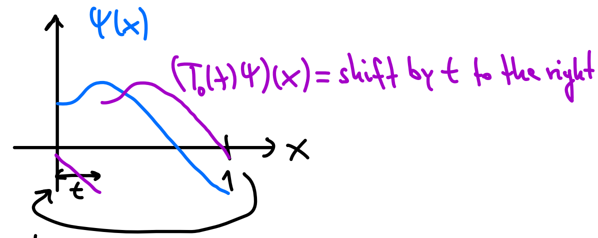
Prof. Sören Petrat Nov. 5, 2021

Example: Translation operator on $L^2([0,1])$

We want to define translations as a unitary group:

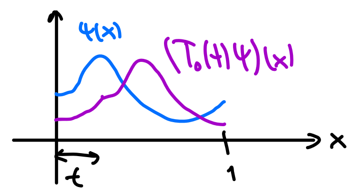


$\Rightarrow$ For $t \in [0,1)$, an obvious translation operator is

$$(T_0(t)\psi)(x) = \begin{cases} \psi(x-t) & \text{if } x-t \in [0,1], \\ \psi(\underbrace{x-t+1}_{\in [0,1) \text{ here}}) & \text{if } x-t < 0. \end{cases}$$

How can $-i\frac{d}{dx}$ be a generator here?

$\hookrightarrow$ $-i\frac{d}{dx}(T_0(t)\psi)(x)$ only exists in L^2 if $\psi(0)=\psi(1)$:



More generally, let us define translations for $t \in [0,1)$ as

$$(T_\theta(t)\psi)(x) = \begin{cases} \psi(x-t) & \text{if } x-t \in [0,1] \\ e^{i\theta}\psi(x-t+1) & \text{if } x-t < 0 \end{cases}, \text{ for any phase factor } \theta \in [0,2\pi).$$

$T_\theta(t)$ is clearly unitary, and we define T_θ for all $t \in \mathbb{R}$ by the group property (e.g. if $t,s \in [0,1)$, then T_θ is def. on $[0,2)$ by $T_\theta(t+s) = T_\theta(t)T_\theta(s)$).

Now: $T_\theta \neq T_{\theta'}$ for $\theta \neq \theta'$, so according to Proposition 3.33 iv) their generators must be different.

Consider $D_\theta : \mathcal{D}(D_\theta) \to L^2([0,1])$, $\psi \mapsto -i\frac{d}{dx}\psi$, with domain

$$\mathcal{D}(D_\theta) = \{\psi \in \underbrace{H^1([0,1])}: e^{i\theta}\underbrace{\psi(1)} = \underbrace{\psi(0)}\}.$$

$H^1([0,1]) := \{\psi \in L^2([0,1]) : \exists \varphi \in H^1(\mathbb{R})$ s.t. $\varphi|_{[0,1]} = \psi\}$

indeed ψ is defined pointwise because of the Sobolev lemma: $H^1(\mathbb{R}) \subset C(\mathbb{R})$.

Then indeed D_θ is the generator of T_θ.

Consistency check: For $\psi, \varphi \in H^1([0,1])$ we find:

$$\langle \psi, -i\frac{d}{dx}\varphi \rangle = \int_0^1 \overline{\psi(x)} \left(-i\frac{d}{dx}\varphi(x)\right)dx$$

integration by parts

$$= -i\left(\overline{\psi(1)}\varphi(1) - \overline{\psi(0)}\varphi(0)\right) + \langle -i\frac{d}{dx}\psi, \varphi \rangle$$

Therefore:

- $-i\frac{d}{dx}$ not symmetric on $D_{max} = H^1([0,1])$ (boundary terms do not vanish), so $-i\frac{d}{dx}$ with domain D_{max} is not a generator
- On D_θ and $D_{min} := \{\psi \in H^1([0,1]) : \psi(0) = 0 = \psi(1)\}$, $-i\frac{d}{dx}$ is symmetric (boundary terms vanish). But on D_{min} it is not a generator, so symmetry is a necessary but not sufficient condition.

  D_{min} is not invariant under any T_θ

Conclusions:

- In applications, we often know operators formally ($-i\frac{d}{dx}$ in this example), but we might not know the domain. It is usually most convenient to choose the domain small (nice regular fct.s), but if we choose it too small (D_{min} in this example), we might not get a generator. Then we try to enlarge the domain, but if we enlarge it too much (D_{max} in this example), we again might not get a generator. Note that enlarging the domain does not necessarily lead to a unique generator (many possibilities D_θ in this example).

- Symmetry is a necessary but not sufficient condition for generators.
  The right class of operators are self-adjoint operators, which we consider next.

3.3 Self-adjoint Operators

We consider bounded operators first.

Recall the general definition of the adjoint (here for normed spaces):

Definition 3.38: Let V and W be normed spaces and $A \in \mathcal{L}(V,W)$. Then the adjoint operator $A': W' \to V'$ (where V' and W' are the dual spaces of V and W) is defined by

$$A'(w')(v) = w'(Av) \quad \forall v \in V.$$

Note:
- For any normed space V, the dual space V' is a Banach space (even if V is not). This is so because elements of V' are continuous, i.e., bounded operators $V \to \mathbb{C}$, and $\mathbb{C}$ is complete (cf. Proposition 3.17).
- $A' \in \mathcal{L}(W', V')$ due to the definition
- With the Hahn-Banach theorem one can show that in fact $\|A'\|_{\mathcal{L}(W',V')} = \|A\|_{\mathcal{L}(V,W)}$.

Hilbert spaces are particularly nice because $\mathcal{H}'$ is isometrically isomorphic to $\mathcal{H}$. (We already noted that $L^p \cong (L^q)'$, $\frac{1}{p} + \frac{1}{q} = 1$ in HW 3, so $L^2 \cong (L^2)'$.) So for $A \in \mathcal{L}(\mathcal{H})$, we would like to identify the operator $A' \in \mathcal{L}(\mathcal{H}')$ with an operator on $\mathcal{H}$. Let us first establish this connection; then we can introduce the notion of self-adjointness.

The key theorem is:

Theorem 3.39: The Riesz Representation Theorem

Let $\mathcal{H}$ be a Hilbert space and $T \in \mathcal{H}'$. Then there is a unique $\psi_T \in \mathcal{H}$ s.t.

$$T(\varphi) = \langle \psi_T, \varphi \rangle_{\mathcal{H}} \quad \forall \varphi \in \mathcal{H}.$$

Proof:

First, if $T(\varphi) = 0$ $\forall \varphi \in \mathcal{H}$, then $T = 0$ and $\psi_T = 0$ is the unique vector in the theorem. Otherwise, we want to show that T is the projection on the one-dimensional subspace spanned by some ψ_T.

So if we consider the kernel $M = \ker(T)$ $(:= \{\varphi \in \mathcal{H} : T(\varphi) = 0\})$, a closed subspace of $\mathcal{H}$ (since T is continuous), we need to show that $M^\perp$ is one-dimensional. If $M = \mathcal{H}$, i.e., $\dim M^\perp = 0$, then $\psi_T = 0$, so let us assume $\dim M^\perp > 0$.

But this follows directly from linearity: Let $\psi, \tilde{\psi} \in M^\perp \setminus \{0\}$. Then for $\alpha \in \mathbb{C}$,

$$T(\psi - \alpha\tilde{\psi}) = T(\psi) - \alpha T(\tilde{\psi}), \text{ so for } \alpha = \frac{T(\psi)}{T(\tilde{\psi})}, \text{ we have } T(\psi - \alpha\tilde{\psi}) = 0, \text{ i.e.,}$$

$$\psi - \alpha\tilde{\psi} \in M, \text{ so } \psi - \alpha\tilde{\psi} \in M \cap M^\perp = \{0\} \text{ and } \psi = \alpha\tilde{\psi}.$$

unique: $\frac{\langle \alpha\tilde{\psi}, \varphi \rangle}{\|\alpha\tilde{\psi}\|^2}\alpha\tilde{\psi} = \frac{\langle \tilde{\psi}, \varphi \rangle}{\|\tilde{\psi}\|^2}\tilde{\psi}$ $\forall \alpha \in \mathbb{C}, \alpha \neq 0.$

Now we can uniquely decompose (with Theorem 3.15) any $\varphi = \varphi_M + \varphi_{M^\perp} = \varphi_M + \frac{\langle \tilde{\psi}, \varphi \rangle}{\|\tilde{\psi}\|^2}\tilde{\psi}$ for any $\tilde{\psi} \in M^\perp \setminus \{0\}$, and thus

$$T(\varphi) = T\left(\varphi_M + \frac{\langle \tilde{\psi}, \varphi \rangle}{\|\tilde{\psi}\|^2}\tilde{\psi}\right) \overset{T(\varphi_M)=0}{=} \frac{\langle \tilde{\psi}, \varphi \rangle}{\|\tilde{\psi}\|^2} T(\tilde{\psi}) = \left\langle \frac{\overline{T(\tilde{\psi})}}{\|\tilde{\psi}\|^2}\tilde{\psi}, \varphi \right\rangle, \text{ i.e., } \psi_T = \frac{\overline{T(\tilde{\psi})}}{\|\tilde{\psi}\|^2}\tilde{\psi}. \quad \square$$