Operations Research, Jacobs University, Fall 2021 — Session 5

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2.3 The Simplex Method

Last time:
- For a standard form LP problem we need to check only basic feasible solutions.
- We can find basic solutions with Gaussian elimination.

But: It would be inconvenient to test all of them (or too computationally expensive for large LP problems).

> The following simplex algorithm is faster:
>
> (i) Start with any basic solution.
>
> (ii) Swap one basic variable ("leaving variable") for another variable ("entering variable") s.t. objective function improves the most.
>
> (iii) If this cannot be done, stop; otherwise repeat.

Let us apply this to the previous example.

We write it as a "simplex tableau":

$$
\begin{array}{c|ccccccc|c}
 & x_1 & x_2 & u & v & s_1 & s_2 & s_3 & \\
\hline
 & 1 & 1 & -1 & 1 & 0 & 0 & 0 & 1 \\
A & 2 & -1 & -2 & 2 & 1 & 0 & 0 & 5 \quad b \\
 & 1 & -1 & 0 & 0 & 0 & 1 & 0 & 4 \\
 & 0 & 1 & 1 & -1 & 0 & 0 & 1 & 5 \\
\hline
c^T & -1 & -2 & -3 & 3 & 0 & 0 & 0 & z
\end{array}
$$

(i) We need to find a basic solution.

Let us choose x_1, s_1, s_2, s_3 columns as our pivot columns.

$\Rightarrow$ We need to eliminate ◯ entries.

now these four columns are the pivot columns

$$\Rightarrow \quad \begin{array}{r} \\ \\ -2R_1+R_2 \to R_2: \\ -R_1+R_3 \to R_3: \\ \\ R_1+R_5 \to R_5: \end{array} \begin{array}{ccccccc|c} x_1 & x_2 & u & v & s_1 & s_2 & s_3 & \\ 1 & 1 & -1 & 1 & 0 & 0 & 0 & 1 \\ 0 & -3 & 0 & 0 & 1 & 0 & 0 & 3 \\ 0 & -2 & 1 & -1 & 0 & 1 & 0 & 3 \\ 0 & 1 & 1 & -1 & 0 & 0 & 1 & 5 \\ \hline 0 & -1 & -4 & 4 & 0 & 0 & 0 & z+1 \end{array}$$

A basic feasible solution is: $x_1 = 1$, $x_2 = 0$, $u = 0$, $v = 0$, $s_1 = 3$, $s_2 = 3$, $s_3 = 5$.

recall how we got particular solutions (Session 3)

There, $c^T x = 0 \Rightarrow z + 1 = 0$, i.e., $z = -1$

c^T is the last row (left-hand side)

$$x = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 3 \\ 3 \\ 5 \end{pmatrix}$$

Note: We make one more simplification in the notation:

$$\begin{array}{ccccccc|c} x_1 & x_2 & u & v & s_1 & s_2 & s_3 & \\ 1 & 1 & -1 & 1 & 0 & 0 & 0 & 1 \\ 0 & -3 & 0 & 0 & 1 & 0 & 0 & 3 \\ 0 & -2 & 1 & -1 & 0 & 1 & 0 & 3 \\ 0 & 1 & 1 & -1 & 0 & 0 & 1 & 5 \\ \hline 0 & -1 & -4 & 4 & 0 & 0 & 0 & 1 \end{array}$$

we delete the z here; then this number is equal to $-z$ at the basic solution.

this tracks $-z$, i.e., this shall be maximized

(ii) Entry variable: Go along direction that improves objective function the most.

new entry variable

$$\begin{array}{ccccccc|c} x_1 & x_2 & u & v & s_1 & s_2 & s_3 & \\ \hline 1 & 1 & -1 & 1 & 0 & 0 & 0 & 1 \\ 0 & -3 & 0 & 0 & 1 & 0 & 0 & 3 \\ 0 & -2 & 1 & -1 & 0 & 1 & 0 & 3 \\ 0 & 1 & 1 & -1 & 0 & 0 & 1 & 5 \\ \hline 0 & -1 & -4 & 4 & 0 & 0 & 0 & 1 \end{array}$$

Geometrically: Go along direction where the slope is most negative.

Choose the column variable where the entry in the last row is the most negative. (If all entries are positive, we are done, because z cannot be decreased further.)

Leaving variable: Where we put the new pivot.

where the pivot in R_4 used to be

Let's test: Take pivot (in column u) in R_4 (row 4), i.e., s_3 as leaving variable

need to be zero

$$\begin{array}{lccccccc|c} R_3: & 0 & -2 & 1 & -1 & 0 & 1 & 0 & 3 \\ R_4: & 0 & 1 & 1 & -1 & 0 & 0 & 1 & 5 \end{array}$$

$R_3 - R_4 \to R_3$:

$$\begin{array}{ccccccc|c} 0 & -3 & 0 & 0 & 0 & 1 & -1 & -2 \\ 0 & 1 & 1 & -1 & 0 & 0 & 1 & 5 \end{array}$$

does not work, because here $s_2 = -2$ (but we need $s_2 \geq 0$)

note: $x_2 = v = s_3 = 0$ at new basic sol.

$\Rightarrow$ Need to choose another leaving variable (next time).