Operations Research, Jacobs University, Fall 2021 — Session 8

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## 2.4 The Dual LP Problem

Let us consider another example:

A factory produces cars ($x_1$) and trucks ($x_2$). We aim at maximizing the profit $z = 3x_1 + 2x_2$, subject to the constraints

profit per car — profit per truck → minimize $z' = -3x_1 - 2x_2$

$5x_1 \le 100$ (car assembly; need 5 h per car; 100 h available) (1)

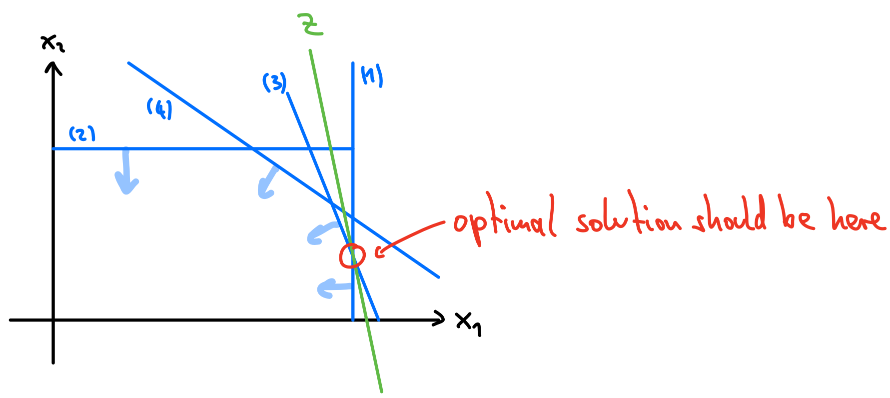
$10x_2 \le 100$ (truck assembly) (2)

$4x_1 + 3x_2 \le 100$ (metal stamping) (3)

$3x_1 + 5x_2 \le 100$ (engine assembly) (4)

$x_1, x_2 \ge 0$

Graphically (rough sketch):




Now we would compute the solution via the simplex method:

| $x_1$ | $x_2$ | $s_1$ | $s_2$ | $s_3$ | $s_4$ | |
|---|---|---|---|---|---|---|
| 5 | 0 | 1 | 0 | 0 | 0 | 100 |
| 0 | 10 | 0 | 1 | 0 | 0 | 100 |
| 4 | 3 | 0 | 0 | 1 | 0 | 100 |
| 3 | 5 | 0 | 0 | 0 | 1 | 100 |
| -3 | -2 | 0 | 0 | 0 | 0 | 0 |

entry variable: $x_1$

leaving variable: $s_1$

| | $x_1$ | $x_2$ | $s_1$ | $s_2$ | $s_3$ | $s_4$ | |
|---|---|---|---|---|---|---|---|
| $R_1/5 \to R_1$: | 1 | 0 | $\frac{1}{5}$ | 0 | 0 | 0 | 20 |
| | 0 | 10 | 0 | 1 | 0 | 0 | 100 |
| $-\frac{4}{5}R_1+R_3 \to R_3$: | 0 | 3 | $-\frac{4}{5}$ | 0 | 1 | 0 | 20 |
| $-\frac{3}{5}R_1+R_4 \to R_4$: | 0 | 5 | $-\frac{3}{5}$ | 0 | 0 | 1 | 40 |
| $\frac{3}{5}R_1+R_5 \to R_5$: | 0 | -2 | $\frac{2}{5}$ | 0 | 0 | 0 | 60 |

entry variable: $x_2$

leaving variable: $s_3$

| | $x_1$ | $x_2$ | $s_1$ | $s_2$ | $s_3$ | $s_4$ | |
|---|---|---|---|---|---|---|---|
| | 1 | 0 | $\frac{1}{5}$ | 0 | 0 | 0 | 20 |
| $-\frac{10}{3}R_3+R_2 \to R_2$: | 0 | 0 | $\frac{40}{15}$ | 1 | $-\frac{10}{3}$ | 0 | $\frac{100}{3}$ |
| $R_3/3 \to R_3$: | 0 | 1 | $-\frac{4}{15}$ | 0 | $\frac{1}{3}$ | 0 | $\frac{20}{3}$ |
| $-\frac{5}{3}R_3+R_4 \to R_4$: | 0 | 0 | $\frac{11}{15}$ | 0 | $-\frac{5}{3}$ | 1 | $\frac{20}{3}$ |
| $\frac{2}{3}R_3+R_5 \to R_5$: | 0 | 0 | $\frac{1}{15}$ | 0 | $\frac{2}{3}$ | 0 | $\frac{220}{3}$ |

Done!

=> Optimal solution: $x_1 = 20$, $x_2 = \frac{20}{3}$, $s_2 = \frac{100}{3}$, $s_4 = \frac{20}{3}$, $s_1 = 0$, $s_3 = 0$, with $z' = -\frac{220}{3}$

(meaning profit $z = -z' = \frac{220}{3}$)

Now note:

- Constraints 1 and 3 hold with equality (no slack: $s_1 = 0$, $s_3 = 0$). These are the binding constraints.
- Constraints 2 and 4 are non-binding.

Generally:

- We have $n$ physical variables, $m$ slack variables (constraints); here: $n=2$, $m=4$.
- Unless there is redundancy, $m$ variables are basic.
- Typically, all physical variables are basic, so also $m-n$ slack variables are basic.
- Thus, $n$ slack variables are typically non-basic: These correspond to the binding constraints.

Knowing our optimal solution, we could forget about the non-binding constraints, i.e., we can write our solution in the following way:

(1) and (3) are the binding constraints

$$\text{let } \tilde{A} = \begin{array}{cc} x_1 & x_2 \\ \end{array} \begin{pmatrix} 5 & 0 \\ 4 & 3 \end{pmatrix} \begin{array}{l} \leftarrow \text{constraint (1)} \\ \leftarrow \text{constraint (3)} \end{array} \qquad \tilde{b} = \begin{pmatrix} 100 \\ 100 \end{pmatrix}, \text{ want } \tilde{A}x = \tilde{b}.$$

Since $\tilde{A}$ is now a square $(n \times n)$ matrix with full rank, it can be inverted: $x = \tilde{A}^{-1}\tilde{b}$.

$$\Rightarrow z' = c^T x = \underbrace{c^T \tilde{A}^{-1}}_{=:\tilde{y}^T} \tilde{b} = y^T b \quad , \text{ where } \tilde{y} \in \mathbb{R}^2, y \in \mathbb{R}^4 \text{ filled up with zeros.}$$

i.e., here $y = \begin{pmatrix} \vdots \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

Note: For $2\times2$ matrices, the formula for the inverse is: $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

Here concretely: $\tilde{y}^T = c^T \tilde{A}^{-1} = (-3,-2)\begin{pmatrix} 5 & 0 \\ 4 & 3 \end{pmatrix}^{-1} = (-3,-2)\,\frac{1}{15}\begin{pmatrix} 3 & 0 \\ -4 & 5 \end{pmatrix}$

$$= \frac{1}{15}(-1,-10). \qquad \left(\Rightarrow y^T = \frac{1}{15}(-1,0,-10,0)\right)$$

Test: $\tilde{y}^T\tilde{b} = \frac{1}{15}(-1,-10)\begin{pmatrix} 100 \\ 100 \end{pmatrix} = \frac{-1100}{15} = \frac{-220}{3} \quad (= z') \ \checkmark$

Now: Change capacities $b$ by a small amount; small meaning the binding constraints remain the same.

Then $b \longrightarrow b + \delta$ and we need to solve $x = \tilde{A}^{-1}(\tilde{b} + \tilde{\delta})$.

$$\Rightarrow \text{new profit } z(\delta) = y^T(b+\delta) = \underbrace{y^T b}_{=z(0)} + y^T \delta$$

The $y_1, \dots, y_m$ are called $\boxed{\text{shadow prices}}$. These are the changes of profit per unit of capacity at current operating conditions.

Indeed, the following holds:

**Theorem:** The value of a company in terms of maximal profit from its operation equals the value of all its resources valued at the current shadow prices.

Proof: Set $\delta = -b$, so $z(\delta) = y^T(b-b) = 0$ (no resources, no operations, no profit)

↳ Note: this does not change the binding constraints (even though $\delta = -b$ is large), since we just rescale the feasible region proportionally.

Thus $z(\delta) = z(0) - y^T b = 0$, so $\underbrace{z(0)}_{\text{value of the company}} = \underbrace{y^T b}_{\text{resources valued at shadow prices}}$.

$\square$