For the functions below, analyze the following points:

1. The domain of the function.
2. The y-intercept and x-intercepts (if any) when they are easy to compute.
3. Horizontal asymptotes.
4. Vertical asymptotes.
5. Analysis of the first derivative (intervals where the function is increasing or decreasing, local minima or maxima).
6. Analysis of the second derivative (intervals where the function is concave up or concave down, points of inflection).

Finally, sketch the function. Your drawing does not need to be to scale, but should show all the qualitative features of the graph.

1 (a) $f(x) = \frac{x^2}{4-x^2}$

2 (b) $f(x) = -\ln(x) + \sqrt{x}$

3 (c) $f(x) = 2\mathrm{e}^{-4/x}$

Solution

1 (a) $f(x) = \frac{x^2}{4-x^2}$
We can see that the function is well defined on $\mathbb{R}$ except for $x^2 = 4 \Rightarrow x = \pm 2$
Therefore

$$\text{Dom}[f] = \mathbb{R} \setminus \{-2, 2\}$$

Within this domain, the equality $f(x) = 0$ is only true if $x = 0$, and thus this is the only intercept (x and y).

We also have that

$$\lim_{x \to \pm\infty} \frac{x^2}{4 - x^2} = \lim_{x \to \pm\infty} \frac{1}{\frac{4}{x^2} - 1} = \frac{1}{0 - 1} = -1$$

and so we have a horizontal asymptote at $y = -1$
Given that the denominator of $f(x)$ approaches 0 at $x = \pm 2$ and the numerator is bounded ($= 4$ in fact), we have vertical asymptotes at $x = \pm 2$.

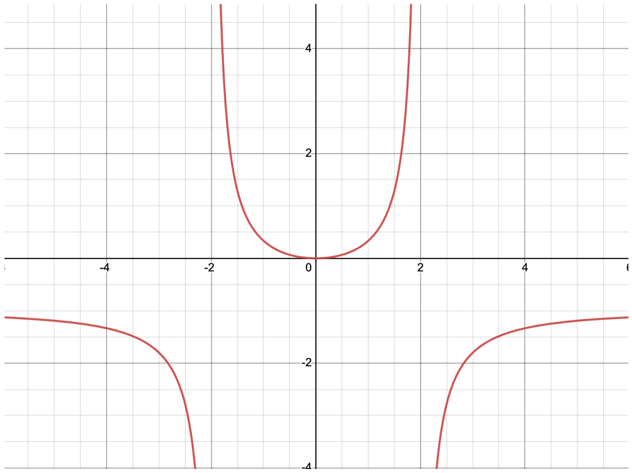
We proceed to calculate the derivative:

$$f'(x) = \frac{(x^2)' \cdot (4 - x^2) - x^2 \cdot (4 - x^2)'}{(4 - x^2)^2} = \frac{8x - 2x^3 - 0 + 2x^3}{(4 - x^2)^2} = \frac{8x}{(4 - x^2)^2}$$

This derivative also has singularities at $x = \pm 2$, and since the denominator is always positive, we can see that the derivative is positive for $x > 0$, negative for $x < 0$, and has a zero at $x = 0$ (all of this in the admissible domain). This illustrates that $f(x)$ has a minimum at $x = 0$, it decreases at $x < 0$, and increases at $x > 0$ with possible discontinuities at $x \pm 2$
Then we analyze the second derivative:

$$\begin{aligned} f''(x) &= \frac{(8x)' \cdot (4 - x^2)^2 - 8x \cdot ((4 - x^2)^2)'}{(4 - x^2)^4} = \frac{8\,(4 - x^2)^2 + 32x^2\,(4 - x^2)}{(4 - x^2)^4} \\ &= \frac{8(4 - x^2) + 32x^2}{(4 - x^2)^3} = \frac{8\,(3x^2 + 4)}{(4 - x^2)^3} \end{aligned}$$

Analyzing this function shows that $f''(x)$ is negative (i.e., f is concave down) in $(-\infty, -2) \bigcup (2, \infty)$ and positive (i.e., f is concave up) in $(-2, 2)$.


4
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-4
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2
4
-2
-4

2 (b) $f(x) = -\ln(x) + \sqrt{x}$

This function is properly defined on $(0, \infty)$, where 0 is not included because of $\ln(x)$, and thus this is the domain of the function.

There are no y intercepts because the domain does not include $y = 0$.
We also have $\lim_{x\to\infty} f(x) \to \infty$ as $\sqrt{x}$ dominates $-\log(x)$ as $x \to \infty$, so no horizontal asymptotes. There is only one vertical asymptote at $x = 0$.

We proceed to calculate the first derivative:

$$f'(x) = -\frac{1}{x} + \frac{1}{2\sqrt{x}}$$

With this we see

$$f'(x) = 0 \Rightarrow 2 = \sqrt{x} \implies x = 4$$

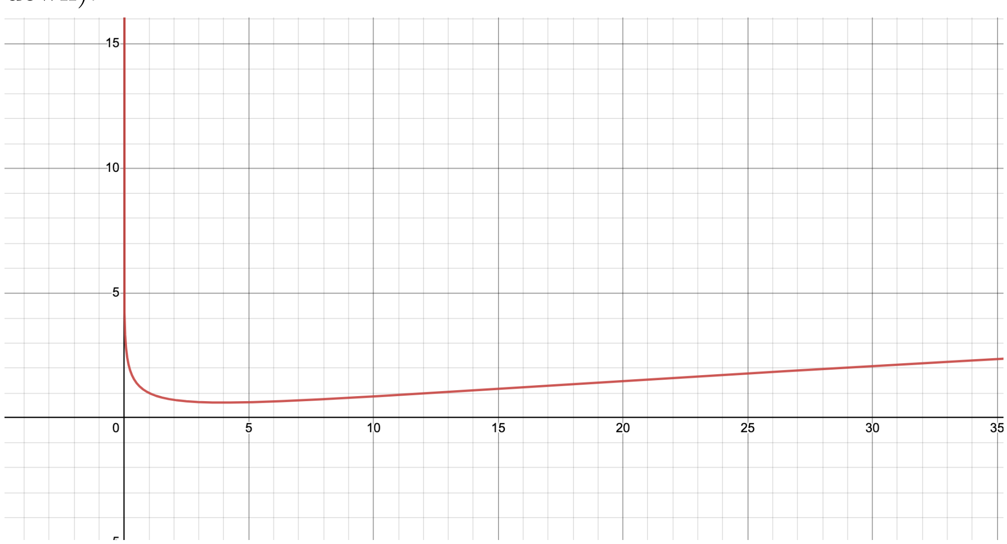
We find $f(4) \approx 0.61$, so f has no x intercepts. We can additionally see that $f'(x) < 0$ for $x < 2$, and $f'(x) > 0$ otherwise.
Lastly, we calculate the second derivative:

$$f''(x) = \frac{1}{x^2} - \frac{1}{4x^{\frac{3}{2}}}$$

And attempting to find a root shows:

$$f''(x) = 0 \Rightarrow 4 = \sqrt{x} \Rightarrow x = 16$$

Thus, $f''(x) > 0$ for $x < 16$ (f is concave up) and $f''(x) > 0$ otherwise (f is concave down).



3 (c) $f(x) = 2e^{-\frac{4}{x}}$

This function is defined everywhere but at $x = 0$, so $\text{Dom}[f] = \mathbb{R} \setminus \{0\}$

Since $\exp : \mathbb{R} \to (0, \infty)$, we can see that there is no x intercept (as $f(x)$ can never equal 0), and we have no y intercept as the function is not defined at $x = 0$.

Taking the limit as $x \to 0$ (from the left) shows that we have a vertical asymptote at this point. We also see that

$$\lim_{x \to \pm\infty} 2e^{-\frac{4}{x}} = 2e^0 = 2,$$

so we have a horizontal asymptote at $y = 2$.

We proceed to calculate the first derivative:

$$f'(x) = 2e^{-\frac{4}{x}} \cdot \left(-\frac{4}{x}\right)' = \frac{8e^{-\frac{4}{x}}}{x^2}$$

It shares the same sign (positive) as the original function since the denominator is always positive (except at $x = 0$ where it is not defined).

Thus, $f(x)$ is always increasing (with a possible discontinuity at $x = 0$).

Lastly, we compute the second derivative:

$$f''(x) = \frac{\left(8e^{-\frac{4}{x}}\right)' \cdot x^2 - (x^2)' \cdot 8e^{-\frac{4}{x}}}{x^4} = \frac{(32 - 16x)\, e^{-\frac{4}{x}}}{x^4}$$

We can see that

$$f''(x) = 0 \Rightarrow 32 = 16x \Rightarrow x = 2$$

and so we know $f(x)$ has an inflection point at $x = 2$. Given that $\frac{e^{-\frac{4}{x}}}{x^4}$ is always positive, we only need to check the sign of $32 - 16x$, which is clearly positive for $x < 2$ and negative for $x > 2$. This results in the following graph: