1 (a) 1. Show that the volume enclosed when revolving the curve $y = f(x)$ - where $f\colon [a,b] \to [0,\infty)$ - about the x-axis in three-dimensional x-y-z space is given by

$$V = \pi \int_a^b f^2(x)\,\mathrm{d}x\,.$$

Hint: Think about the cross-sectional areas and the perfect symmetry when revolving the function around the x-axis.

2. Compute the volume of the solid obtained by revolving the graph of $y = \dfrac{1+x^2}{2}$ on $[0,1]$ about the x-axis.

2 (b) 1. Hook's law states that the force exerted by an ideal spring when extended from its equilibrium position at $x = 0$ to length x is given by

$$F(x) = -k\,x\,,$$

where k is a positive constant characterizing the stiffness of the spring. Compute the work required to expand the spring from its equilibrium position to length ℓ.

2. Show that

$$\int_0^\infty \frac{1}{\sqrt{1+x^4}}\,\mathrm{d}x$$

is convergent.
Hint: There is no elementary way to evaluate this integral. However, to only *test* convergence, you can bound the integrand by a simpler function and use the following fact without proof: Let $f\colon [a,\infty) \to \mathbb{R}$ be a bounded and increasing function. Then $\lim_{x\to\infty} f(x)$ exists. (The whole integral corresponds to the function f here.)

3 (c) One step to deriving Kepler's Orbit is to derive the integral given by:

$$\varphi = \int \frac{\frac{1}{r^2}}{\sqrt{2\,(E - V_{\text{eff}}(r))}}\,\mathrm{d}r, \qquad V_{\text{eff}} = -\frac{1}{r} + \frac{1}{2r^2}$$

Here, most of the constants pertaining to the original equation have been set equal to 1 (except for 'E', which stands for the total energy of the system and should be treated as a constant). Here, r represents the radius of the orbiting object, φ is the angle covered by the object in orbit, and V_{eff} is the effective potential energy we are considering for the system.

1. Complete the square in the denominator and find and use substitution to reformulate the integral and leave it in the form of

$$\varphi = -\int \frac{\mathrm{d}\mu}{\sqrt{1-\mu^2}}$$

2. Apply substitution once again and find an explicit expression for r as a function of φ

Solution

1 (a) 1. Split the volume into thin disks (with the base normal parallel to the OX axis). Each disk has a volume $\pi f(x)^2 \mathrm{d}x$ ($f(x)$ is the base radius). The total volume is then:

$$V = \int \mathrm{d}V = \int_a^b \pi f(x)^2 \, \mathrm{d}x$$

2. Using the formula above:

$$V = \int_0^1 \pi \left(\frac{1+x^2}{2}\right)^2 \mathrm{d}x = \frac{\pi}{4}\int_0^1 \left[1 + 2x^2 + x^4\right] \mathrm{d}x =$$

$$= \frac{\pi}{4}\left(x + \frac{2x^3}{3} + \frac{x^5}{5}\right)\Bigg|_0^1 = \frac{\pi}{4}\left(1 + \frac{2}{3} + \frac{1}{5}\right) = \frac{7\pi}{15}$$

2 (b) 1. Compute the work by using the equations given in the problem statement:

$$\int_0^l kx \, \mathrm{d}x = \frac{kl^2}{2}$$

2. Use suitable estimates:

$$\frac{1}{\sqrt{1}} \geq \frac{1}{\sqrt{1+x^4}} \text{ for } 0 \leq x \leq 1 \text{ and } \frac{1}{\sqrt{x^4}} \geq \frac{1}{\sqrt{1+x^4}} \text{ for } x > 1$$

Now one can bound further:

$$\int_0^\infty \frac{1}{\sqrt{1+x^4}} \mathrm{d}x \leq \int_0^1 \frac{1}{\sqrt{1}} \mathrm{d}x + \int_1^\infty \frac{1}{\sqrt{x^4}} \mathrm{d}x \Rightarrow$$

$$\Rightarrow \int_0^\infty \frac{1}{\sqrt{1+x^4}} \leq \int_0^1 1 \, \mathrm{d}x + \int_1^\infty \frac{1}{x^2} \mathrm{d}x = x\Big|_0^1 - \frac{1}{x}\Big|_1^\infty = 1 - (0-1) = 2 \Rightarrow$$

$$\Rightarrow \int_0^\infty \frac{1}{\sqrt{1+x^4}} \mathrm{d}x \leq 2$$

Thus, $f(x) = \displaystyle\int_0^x \frac{1}{\sqrt{1+x'^4}} \mathrm{d}x'$ is a bounded monotonously increasing function, meaning that $\lim\limits_{x\to\infty} f(x)$ exists, i.e., the integral converges.

3 (c) 1. Use the following substitution: $u = \frac{1}{r}$, $\mathrm{d}u = -\frac{1}{r^2}\mathrm{d}r$.

$$\varphi = \int \frac{\frac{1}{r^2}}{\sqrt{2\left(E + \frac{1}{r} - \frac{1}{2r^2}\right)}} \mathrm{d}r = -\int \frac{\mathrm{d}u}{\sqrt{2\left(E + u - \frac{u^2}{2}\right)}}$$

[Complete the square]

$$= -\int \frac{\mathrm{d}u}{\sqrt{2E + 2u - u^2}} = -\int \frac{\mathrm{d}u}{\sqrt{-\left[(u-1)^2 - (1+2E)\right]}}$$

[Use the substitution: $t = \frac{u-1}{\sqrt{1+2E}}$, $\mathrm{d}t = \frac{1}{\sqrt{1+2E}}\mathrm{d}u$]

$$= -\int \frac{\mathrm{d}t}{\sqrt{-(t^2-1)}} = -\int \frac{\mathrm{d}t}{\sqrt{1-t^2}}$$

2. Another substitution $t = \cos\theta$ yields:

$$\varphi(r) = \arccos(t) + C = \arccos\left(\frac{u-1}{\sqrt{2E-1}}\right) + C = \arccos\left(\frac{\frac{1}{r}-1}{\sqrt{2E-1}}\right) + C$$