In Quantum Mechanics, it is commonly said that angular momentum 'generates' rotations, and in this exercise we will show this statement, starting from definitions:

- We say that H generates U if: $\mathrm{e}^{-i\varphi H} = U$ for some parameter φ.
- The Taylor expansion of a function f around a point a is defined as

$$\mathcal{T}(f) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

1 (a) Find the Taylor expansions around $a = 0$ for the functions: e^x, $\sin(x)$, $\cos(x)$. (From now on, you may assume that the Taylor expansions of these functions are equivalent to the functions themselves.)

2 (b) Given that $\hat{L}_Z = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, compute $(\hat{L}_z)^{2k}$ and $(\hat{L}_z)^{2k+1}$ for all $k \in \mathbb{N}_0$.

3 (c) Using b) and the Taylor expansions from a), show by summing explicitly that

$$\mathrm{e}^{-i\varphi \hat{L}_z} = \mathcal{R}_z(\varphi).$$

Recall: From multiple choice questions we know that rotations around the $z-$axis are given by $\mathcal{R}_z(\varphi) = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Solution

1 (a) Before computing the Taylor expansions, we will first calculate the derivatives of the respective functions:

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^x = \mathrm{e}^x \implies \frac{\mathrm{d}^n}{\mathrm{d}x^n}\mathrm{e}^x = \mathrm{e}^x \\
&\frac{\mathrm{d}}{\mathrm{d}x}\sin x = \cos x \\
&\frac{\mathrm{d}^2}{\mathrm{d}x^2}\sin x = \frac{\mathrm{d}}{\mathrm{d}x}\cos x = -\sin x \\
&\frac{\mathrm{d}^3}{\mathrm{d}x^3}\sin x = \frac{\mathrm{d}}{\mathrm{d}x}(-\sin x) = -\cos x \\
\implies &\frac{\mathrm{d}^{4k}}{\mathrm{d}x^{4k}}\sin x = \sin x \quad \frac{\mathrm{d}^{4k+1}}{\mathrm{d}x^{4k+1}}\sin x = \cos x \quad \frac{\mathrm{d}^{4k+2}}{\mathrm{d}x^{4k+2}}\sin x = -\sin x \quad \frac{\mathrm{d}^{4k+3}}{\mathrm{d}x^{4k+3}}\sin x = -\cos x \\
\text{and } &\frac{\mathrm{d}^{4k}}{\mathrm{d}x^{4k}}\cos x = \cos x \quad \frac{\mathrm{d}^{4k+1}}{\mathrm{d}x^{4k+1}}\cos = -\sin x \quad \frac{\mathrm{d}^{4k+2}}{\mathrm{d}x^{4k+2}}\cos = -\cos x \quad \frac{\mathrm{d}^{4k+3}}{\mathrm{d}x^{4k+3}}\cos = \sin x
\end{aligned}$$

Therefore, the Taylor series are:

$$\mathcal{T}(\mathrm{e}^x) = \sum_{k=0}^{\infty}\frac{x^k}{k!}\left(\frac{\mathrm{d}^k}{\mathrm{d}x^k}\mathrm{e}^x\right)\bigg|_{x=0} = \sum_{k=0}^{\infty}\frac{x^k}{k!}(\mathrm{e}^x)\bigg|_{x=0} = \sum_{k=0}^{\infty}\frac{x^k}{k!}$$

$$\begin{aligned}
\mathcal{T}(\sin x) &= \sum_{k=0}^{\infty}\frac{x^k}{k!}\left(\frac{\mathrm{d}^k}{\mathrm{d}x^k}\sin x\right)\bigg|_{x=0} \\
&= \sum_{k=0}^{\infty}\left\{\frac{x^{4k}}{(4k)!}\sin x + \frac{x^{4k+1}}{(4k+1)!}\cos x + \frac{x^{4k+2}}{(4k+2)!}(-\sin x) + \frac{x^{4k+3}}{(4k+3)!}(-\cos x)\right\}\bigg|_{x=0} \\
&= \sum_{k=0}^{\infty}\left(\frac{x^{4k+1}}{(4k+1)!} - \frac{x^{4k+3}}{(4k+3)!}\right) \\
&= \sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{(2k+1)!} \\
&= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots
\end{aligned}$$

$$\begin{aligned}
\mathcal{T}(\cos x) &= \sum_{k=0}^{\infty} \frac{x^k}{k!}\left(\frac{\mathrm{d}^k}{\mathrm{d}x^k}\cos x\right)\Bigg|_{x=0} \\
&= \sum_{k=0}^{\infty} \left\{ \frac{x^{4k}}{(4k)!}\cos x + \frac{x^{4k+1}}{(4k+1)!}(-\sin x) + \frac{x^{4k+2}}{(4k+2)!}(-\cos x) + \frac{x^{4k+3}}{(4k+3)!}\sin x \right\}\Bigg|_{x=0} \\
&= \sum_{k=0}^{\infty} \left(\frac{x^{4k}}{(4k)!} - \frac{x^{4k+2}}{(4k+2)!} \right) \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} \\
&= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots
\end{aligned}$$

2 (b) $(\hat{L}_Z)^2 = i\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot i\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = -1\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Then $(\hat{L}_Z)^{2k} = ((\hat{L}_Z)^2)^k = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}^k = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

and $(\hat{L}_Z)^{2k+1} = ((\hat{L}_Z)^{2k}) \cdot (\hat{L}_Z) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \hat{L}_z$

This holds for all $k \neq 0$, but for this case $(\hat{L}_z)^0 = \mathbb{1}$ (the identity matrix).

3 (c) We compute:

$$\begin{aligned}
e^{-i\varphi\hat{L}_z} &= \sum_{k=0}^{\infty} \frac{(-i\varphi\hat{L}_z)^k}{k!} = \sum_{k=0}^{\infty} \left(\frac{(-i\varphi\hat{L}_z)^{4k}}{(4k)!} + \frac{(-i\varphi\hat{L}_z)^{4k+1}}{(4k+1)!} + \frac{(-i\varphi\hat{L}_z)^{4k+2}}{(4k+2)!} + \frac{(-i\varphi\hat{L}_z)^{4k+3}}{(4k+3)!} \right) \\
&= \sum_{k=0}^{\infty} \left(\frac{\varphi^{4k}(\hat{L}_z)^{4k}}{(4k)!} + \frac{(-i\varphi^{4k+1})(\hat{L}_z)^{4k+1}}{(4k+1)!} + \frac{(-\varphi^{4k+2})(\hat{L}_z)^{4k+2}}{(4k+2)!} + \frac{i\varphi^{4k+3}(\hat{L}_z)^{4k+3}}{(4k+3)!} \right) \\
&= \sum_{k=0}^{\infty} \left(\frac{\varphi^{4k}}{(4k)!} - \frac{\varphi^{4k+2}}{(4k+2)!} \right) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} - i\sum_{k=0}^{\infty} \left(\frac{\varphi^{4k+1}}{(4k+1)!} - \frac{\varphi^{4k+3}}{(4k+3)!} \right) \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
&= \cos(\varphi)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} - i\sin(\varphi)\begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned}$$