In Quantum Mechanics, it is commonly said that angular momentum 'generates' rotations, and in this exercise we will show this statement, starting from definitions:

- We say that H generates U if: $\mathrm{e}^{-i\varphi H} = U$ for some parameter φ.
- The Taylor expansion of a function f around a point a is defined as

$$\mathcal{T}(f) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

1 (a) Find the Taylor expansions around $a = 0$ for the functions: e^x, $\sin(x)$, $\cos(x)$. (From now on, you may assume that the Taylor expansions of these functions are equivalent to the functions themselves.)

2 (b) Given that $\hat{L}_Z = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, compute $(\hat{L}_z)^{2k}$ and $(\hat{L}_z)^{2k+1}$ for all $k \in \mathbb{N}_0$.

3 (c) Using b) and the Taylor expansions from a), show by summing explicitly that

$$\mathrm{e}^{-i\varphi \hat{L}_z} = \mathcal{R}_z(\varphi).$$

Recall: From multiple choice questions we know that rotations around the $z-$axis are given by $\mathcal{R}_z(\varphi) = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}$.