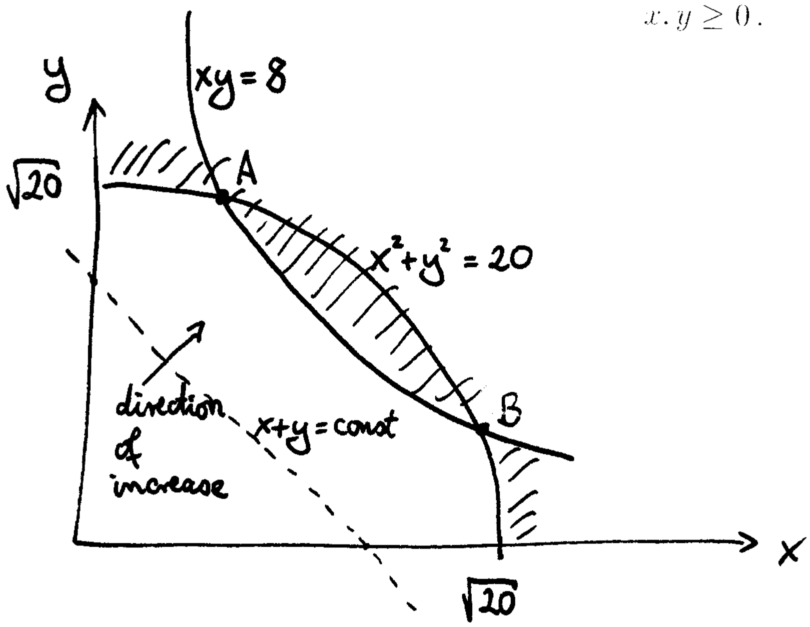
1. Use the graphical method to maximize

$$z = x + y$$

subject to

$$x^2 + y^2 \le 20\,,$$
$$x\,y \le 8\,,$$
$$x, y \ge 0\,.$$

(20)




Since the objective function and the constraints are symmetric under exchange of x and y, the problem is expected to have two optimal solutions at locations A and B, the points of intersection of the two nonlinear constraint boundaries.

To find the coordinates: $xy = 8 \Rightarrow y = \frac{8}{x}$

$$\Rightarrow x^2 + \frac{8^2}{x^2} = 20$$

$$\Rightarrow x^4 - 20x^2 + 64 = 0$$

$$\Rightarrow (x^2-10)^2 = 36 \quad \Rightarrow \quad x^2 - 10 = \pm 6$$

$$\Rightarrow x^2 = 16 \text{ or } x^2 = 4$$

$\Rightarrow$ Max at $A = (2,4)$ and $B = (4,2)$ where $z = 6$.

2. The attached Pyomo notebook shows a solution to the "Diet Problem".

(a) State a concise formulation of the problem *in words*. (No need to repeat the given numerical values of the data.)

(b) Describe the meaning of the dual variables in words.

(c) Which nutrient is supplied at more than its minimal requirement? Explain your answer.

(d) Suggest an additional constraint (or additional constraints) to ensure that the resulting diet is more varied (without changing given data of the problem). State the idea in words and then write out the corresponding lines of Pyomo code.

(5+5+5+5)

(a) Given is a set of foods F. Each food has a unit cost (in c) and specific content of each of the nutrients N, specified in the matrix a.
The task is to find the cheapest diet such that a minimum level of each nutrient, specified in r, is supplied.

(b) There is a dual variable for each nutrient which specifies the marginal increase of the cost of the diet provided the minimum requirement of that nutrient is increased by one unit.

(c) It's Calcium: its shadow price is zero, i.e. the constraint is non-binding.

(d) This can be done in many ways.

<u>Option 1</u>: Require that no food item is used for more than a certain fraction, $\frac{1}{2}$ say, of the total diet.

Pyomo code:

```
def limit_rule (model, j):
    return model.x[j] <= 0.5 *
           sum(model.x[i] for i in F)

model.c2 = Constraint(F, rule = limit_rule)
```

<u>Option 2</u>: Noting that the high milk content in the diet leads to an oversupply of Calcium by a large margin, impose an upper bound constraint on Calcium (or on all nutrients). The code is obvious (just reverse the inequality in the lower bound code).

There are more variations/options, but simply adding additional lower bound constraints may not do the job unless very specifically chosen.

In [1]:
```
from pyomo.environ import *
from pyomo.opt import *
opt = solvers.SolverFactory("glpk")
```

In [2]:
```
F = ['Broccoli', 'Milk', 'Oranges']
N = ['Calcium', 'Vitamin C', 'Water']

c = {'Broccoli':38, 'Milk':10, 'Oranges':27}
r = {'Calcium':1000, 'Vitamin C':90, 'Water':3700}

a = {('Broccoli','Calcium'):47,
     ('Broccoli','Vitamin C'):89,
     ('Broccoli','Water'):91,
     ('Milk','Calcium'):276,
     ('Milk','Vitamin C'):0,
     ('Milk','Water'):87,
     ('Oranges','Calcium'):40,
     ('Oranges','Vitamin C'):53,
     ('Oranges','Water'):87}

model = ConcreteModel()
model.x = Var(F, within=NonNegativeReals)
```

In [3]:
```
def nutrition_rule (model, n):
    return sum(a[i,n]*model.x[i] for i in F) >= r[n]

model.c = Constraint(N, rule=nutrition_rule)

model.z = Objective(expr = sum(c[i]*model.x[i] for i in F),
                    sense=minimize)
```

In [4]:
```
model.dual = Suffix(direction=Suffix.IMPORT)
results = opt.solve(model)
```

In [5]:
```
model.x.get_values()
```

Out[5]:
```
{'Broccoli': 1.01123595505618, 'Milk': 41.4710060699987, 'Oran
ges': 0.0}
```

In [6]:
```
model.z.expr()
```

Out[6]:
```
453.1370269921219
```

In [7]:
```
for n in N:
    print("Shadow price of", n, "is", model.dual[model.c[n]])
```

```
Shadow price of Calcium is 0.0
Shadow price of Vitamin C is 0.309440785225365
Shadow price of Water is 0.114942528735632
```

3. You have decided to take advantage of your holidays and of the recent heavy snowfall to try snowboarding. You hire the equipment at the ski resort. The shop has three different offers: one day hiring for 25€, two days 40€, and four days for 60€.

The weather forecast for the next 8 days is given in the following table where a 1 indicates that conditions are good for snowboarding and a 0 indicates that conditions are bad for snowboarding.

| Day | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| Weather | 1 | 0 | 1 | 1 | 0 | 1 | 0 | 1 |

Use dynamic programming to find out when and for how long to hire a snowboard in order to minimize the total expense assuming you go snowboarding if and only if the weather is good. (20)

At day $i$, let $s_i$ denote the days left on the last hire, and $x_i$ the days on a new hire. Note that a new hire should only be made if $s_i = 0$ AND the weather is good. Let $f_i(s_i, x_i)$ be the cost incurred from day $i$.

Day 8:

$f_8(s_8, x_8)$

| $s_8$ | $x_8=0$ | $x_8=1$ | $f_8^*$ | $x_8^*$ |
|---|---|---|---|---|
| 0 | – | 25 | 25 | 1 |
| 1 | 0 | – | 0 | 0 |
| 2 | 0 | – | 0 | 0 |
| 3 | 0 | – | 0 | 0 |

Day 6:

$f_6(s_6, x_6)$

| $s_6$ | $x_6=0$ | $x_6=1$ | $x_6=2$ | $x_6=4$ | $f_6^*$ | $x_6^*$ |
|---|---|---|---|---|---|---|
| 0 | – | 25+25 | – | 60 | 50 | 1 |
| 1 | 0+25 | | | | 25 | 0 |
| 2 | 0+25 | | | | 25 | 0 |
| 3 | 0+0 | | | | 0 | 0 |

Day 4:

$f_4(s_4, x_4)$

| $s_4$ | $x_4=0$ | $x_4=1$ | $x_4=2$ | $x_4=4$ | $f_4^*$ | $x_4^*$ |
|---|---|---|---|---|---|---|
| 0 | – | 25+50 | – | 60+25 | 75 | 1 |
| 1 | 0+50 | | | | 50 | 0 |
| 2 | 0+50 | | | | 50 | 0 |
| 3 | 0+25 | | | | 25 | 0 |

Day 3:

$f_3(s_3, x_3)$

| $s_3$ | $x_3=0$ | $x_3=1$ | $x_3=2$ | $x_3=4$ | $f_3^*$ | $x_3^*$ |
|---|---|---|---|---|---|---|
| 0 | – | 25+75 | 40+50 | 60+25 | 85 | 4 |
| 1 | 0+75 | | | | 75 | 0 |
| 2 | 0+50 | | | | 50 | 0 |
| 3 | 0+50 | | | | 50 | 0 |

Day 1:

$f_1(s_1, x_1)$

| $s_1$ | $x_1=0$ | $x_1=1$ | $x_1=2$ | $x_1=4$ | $f_1^*$ | $x_1^*$ |
|---|---|---|---|---|---|---|
| 0 | – | 25+85 | – | 60+50 | 110 | 1 or 4 |

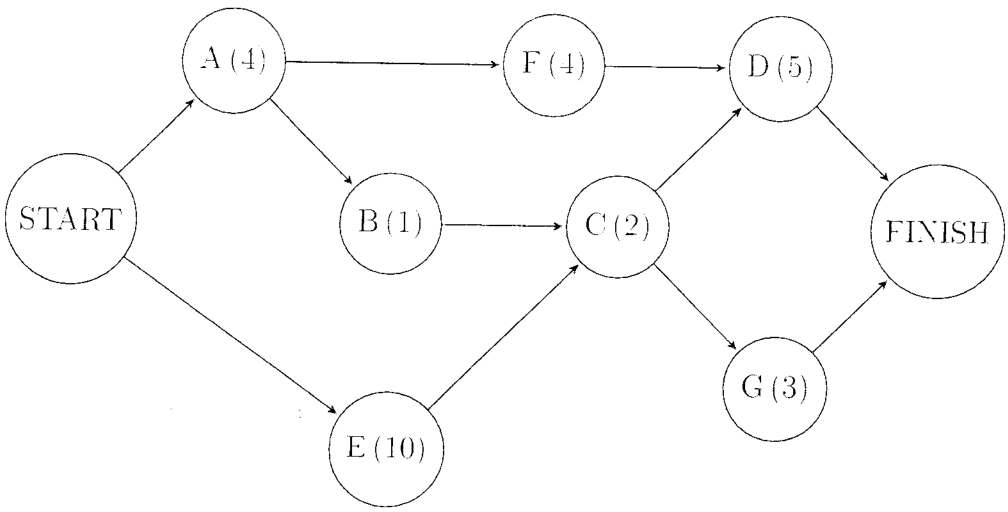
So there are two optimal schedules:

(a) Hire for 1 day on Day 1, for 4 days on Day 3, for 1 day on Day 8

(b) Hire for 4 days on Day 1, for 1 day on Day 6 and Day 8.

Total cost: € 110.

4. A project consists of activities A, ..., G which have the following dependencies:




The time (in days) that each step takes is indicated in parentheses. Each day that the project is being worked on (independent of how many activities are done in parallel) costs 1 000 €. Furthermore, special machinery must be rented from the beginning of task A up to the end of task G at a cost of 5 000 € per day.

(a) Formulate a linear program that minimizes the total cost of the project. You do not need to solve it, but you are required to state a complete mathematical formulation.

(b) In this case, the problem is simple enough to analyze it "by hand". Find the optimal solution.

(10+10)

(a) Let $t_i$ be the starting time of task $i \in \{\text{START}, A, B, \ldots, G, \text{FINISH}\}$.

Fix $t_{START} = 0$. Let $d_i$ be the duration of task $i$. Then:

$$\text{Minimize} \quad t_{FINISH} \cdot 1000 + (t_G + 3 - t_A) \cdot 5000$$

subject to $\quad t_j \geq t_i + d_i \quad$ if task $j$ depends on task $i$.

(b) The critical path is clearly

START → E → C → D → FINISH (length 17)

The expensive path is

A → B → C → G (length 10)

Extending the expensive path by one day costs 5000 €, extending the critical path by one day costs 1000 €.

Since D depends on F and C, it is necessary to extend either the critical or the expensive path by one day, so the choice is to start D late.

$\Rightarrow$ Total cost $= 10 \cdot 5000 + 18 \cdot 1000 = 68000$

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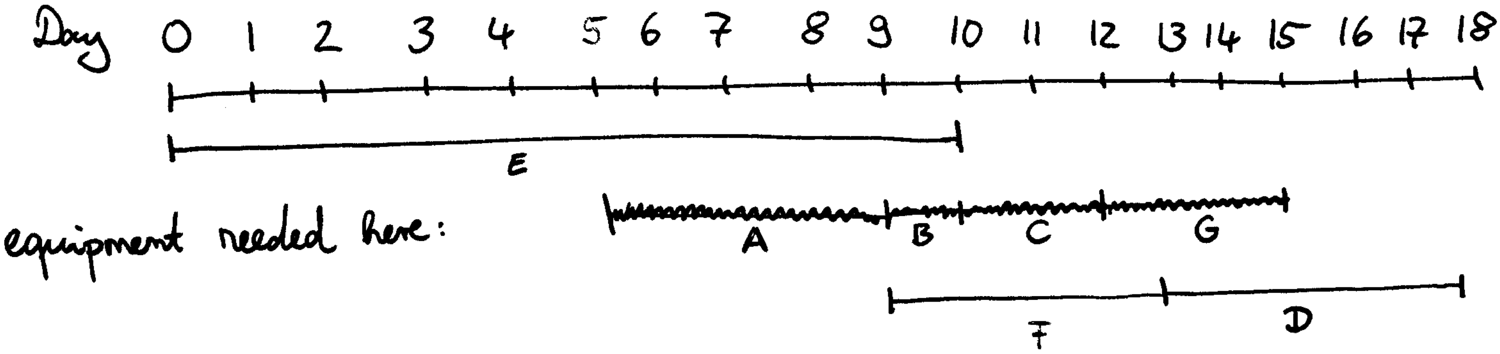
Optimal time-line (not required for full score):



5. Consider the Goferbroke Co. problem discussed in class with the following modified payoff table:

| Alternative | State of Nature Oil | Dry |
|---|---|---|
| Drill | 60 | −15 |
| Sell | 10 | 10 |
| Prior Probability | 1/3 | 2/3 |

The Goferbroke Co. has to decide whether to drill for oil or to sell the land.

(a) Compute the expected payoffs for both alternatives. What decision should be taken based on a risk-neutral analysis?

(b) A company offers exploration which returns the true state of nature in 2/3 of cases and gives a false result in 1/3 of cases. Compute the probability of obtaining a favorable exploration result (i.e., one that indicates presence of oil), and the probability of finding a non-favorable exploration result.

(c) How much should you be willing to pay for these services?

(d) How do you decide upon drilling or selling in response to the exploration result?

(5+5+5+5)

(a) For "Drill": $EP = \frac{1}{3}\cdot 60 - \frac{2}{3}15 = 10$

For "Sell": $EP = 10$

So, on a risk-neutral analysis, both alternatives are of equal value.

(b) Let F be the event of a favorable exploration response,
U " " " " an unfavorable " "

$P(F) = P(F \mid Oil)\, P(Oil) + P(F \mid Dry)\, P(Dry)$

$= \frac{2}{3}\frac{1}{3} + \frac{1}{3}\frac{2}{3} = \frac{4}{9}$



$P(U) = 1 - P(F) = \frac{5}{9}$

(c+d) $P(\text{Oil} \mid F) = \dfrac{P(F \mid \text{Oil})\, P(\text{Oil})}{P(F)} = \dfrac{\frac{2}{3} \cdot \frac{1}{3}}{\frac{4}{9}} = \frac{1}{2}$

$P(\text{Dry} \mid F) = 1 - P(\text{Oil} \mid F) = \frac{1}{2}$

So if exploration is favorable, the expected payoffs are:

For "Drill": $EP = \frac{1}{2} \cdot 60 - \frac{1}{2} \cdot 15 = \frac{45}{2} = 22.5$

For "Sell": $EP = 10$

In this case, "Drill" should be chosen.

If exploration is unfavorable, "Sell" will become the best option as the expected payoff from "Drill" will go down from part (a) above.

$\Rightarrow$ Total expected payoff is $P(F) \cdot \frac{45}{2} + P(U) \cdot 10 = \frac{4}{9} \cdot \frac{45}{2} + \frac{5}{9} \cdot 10$

$= \frac{140}{9}$

$\Rightarrow EVE = \frac{140}{9} - 10 = \frac{50}{9} = 5.55\ldots$

- So the exploration should cost no more than about 5.5 units of money. If this is the case, explore. Otherwise don't explore.
- A favorable exploration should be followed by drilling, an unfavorable by selling.