1. (a) Use the graphical method to maximize

$$z = 2\,x + 3\,y$$

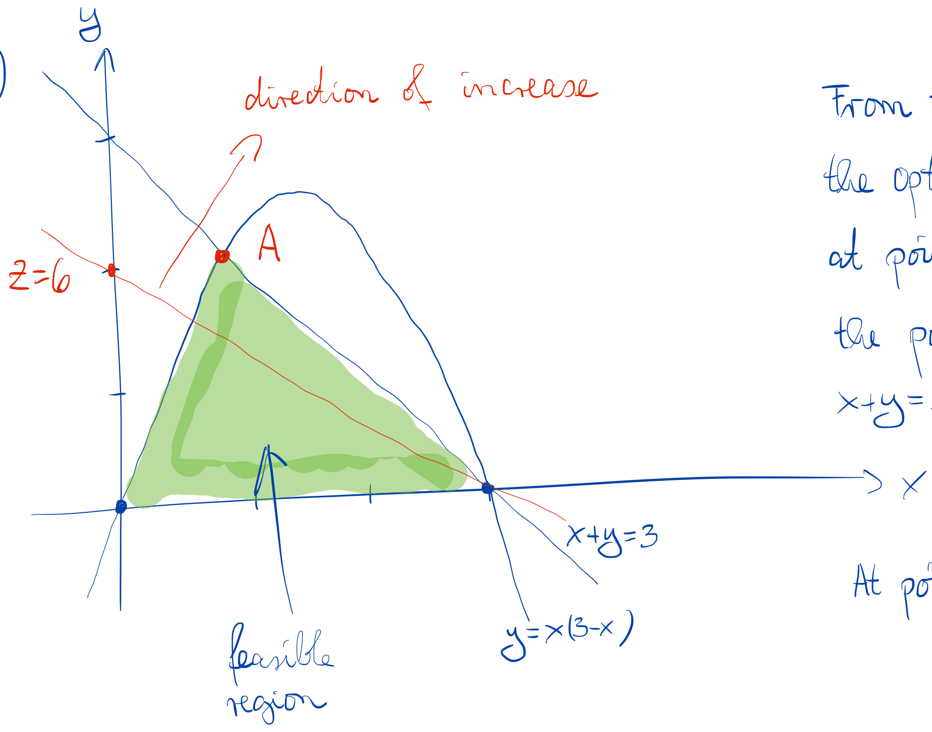
subject to

$$\begin{aligned} x + y &\leq 3\,, \\ 3\,x - x^2 - y &\geq 0\,, \quad \text{or: } y \leq x\,(3-x) \\ x, y &\geq 0\,. \end{aligned}$$

(b) Is this a convex optimization problem? What can you conclude for this problem if convexity was/is satisfied?

(10+5)

(a)



From the sketch, it is clear that the optimal value of z occurs at point A = (1,2) which is the point of intersection of x+y=3 and y= x(3-x).

At point A, $z = 2\cdot 1 + 3\cdot 2 = 8$

(b) The objective function is linear, thus convex. The feasible region is convex, so the problem is convex.

This means it does not have local maxima other than the global maximum at point A.

2. The Chemtails Supply Co. produces scientific laboratory equipment. Production is done by three teams who each need a given amount of time for each unit produced. The unit profit for each product is known and it is assumed that all units that are produced will be sold. Each team has a certain number of hours available. Further, the company has long-term supply contracts for two of their products, the microscope and the Bunsen burner, that require a certain minimum number of units per week to be produced.

   The weekly production schedule has been planned using the Pyomo notebook attached. Given the data and the output from this notebook, answer the following questions.

   (a) Marketing suggests a discount campaign, lowering the price of a thermometer by €1. How will this affect profits on the given weekly production schedule?

   (b) A member of Team 1 is sick and misses one day (8 hours) of work. Estimate the decrease in profit if (i) no other worker can replace her or (ii) a worker of Team 2 can take over her work or (iii) only a worker of Team 3 can take over her work.

   (c) Management would like to increase production in the short term to meet surging demand and decides to offer workers an overtime bonus of half of the marginal increase in profits generated by extra work. Compute the bonus on the hourly wage for each of the teams.

   (d) Controlling are reviewing the long-term contracts. Give a recommendation toward a possible re-negotiation of the terms of these contracts.

   (e) Sales are finding out that they cannot sell more than 100 safety goggles per week at full price. Any additional pairs need to be discounted by €3. Suggest changes to the Pyomo code that model this additional constraint.

(5+5+5+5+5)

(a) Since about 104 thermometers are produced (see Pyomo code), the maximal loss is $104 \cdot 1 = 104$ Euros. It may be less if the optimal solution jumps to another set of basic variables.

(b) The shadow price of the work of Team 1 is 12 €/h, so (i) the loss in profits would be 96 €. If (ii) a member of Team 2 can take over, we have to take Team 2's shadow price for a loss of $3.25 \cdot 8 = 26$ €. If (iii) only a member of Team 3 can take over, that would not help because Team 3's shadow price is higher, so the loss is 96€ as for (i).

(c) Take half the shadow prices, so 6€/h, 1.625€/h, 13.035€/h for Team 1, 2, 3, respectively.

(d) The minimum supply constraint of the microscope has a shadow price of −1.40€, so for each unit less required, profits will go up by 1.40€. So the recommendation is:

- either lower the required number of microscopes until the shadow price of this constraint drops to zero,
- or raise the price of a microscope by 1.40€

(e)
- add a new decision variable "discount goggles" with same work requirement as "safety goggles" but a profit contribution of only 8 (instead of 11).

- add the constraint

  model.goggles = Constraint(expr = model.x['safety goggles'] ≤ 100)

In [1]:
```
from pyomo.environ import *
from pyomo.opt import *
opt = solvers.SolverFactory("glpk")
```

In [2]:
```
t = {'Team 1': 100, 'Team 2': 60, 'Team 3': 80}                                    # [Hours]
p = {'Microscope': 28, 'Thermometer': 5, 'Safety Goggles': 11, 'Bunsen Burner': 13} # [EUR]
r = {'Microscope': 40, 'Thermometer': 0, 'Safety Goggles': 0, 'Bunsen Burner': 20}  # [Units]

w = {('Team 1', 'Microscope'):     0.78, # [Hours]
     ('Team 1', 'Thermometer'):    0.30,
     ('Team 1', 'Safety Goggles'): 0.33,
     ('Team 1', 'Bunsen Burner'):  0,
     ('Team 2', 'Microscope'):     0.15,
     ('Team 2', 'Thermometer'):    0.43,
     ('Team 2', 'Safety Goggles'): 0,
     ('Team 2', 'Bunsen Burner'):  0.23,
     ('Team 3', 'Microscope'):     0.75,
     ('Team 3', 'Thermometer'):    0,
     ('Team 3', 'Safety Goggles'): 0.27,
     ('Team 3', 'Bunsen Burner'):  0.47}

P = list(p.keys())
T = list(t.keys())

model = ConcreteModel()
model.x = Var(P, within=NonNegativeReals)

def profit_rule(model):
    return sum(p[j]*model.x[j] for j in P)
model.profit = Objective(rule=profit_rule, sense=maximize)

def capacity_rule(model, i):
    return sum(w[i,j]*model.x[j] for j in P) <= t[i]
model.capacity_constraint = Constraint(T, rule=capacity_rule)

def fixed_contract_rule(model, j):
    return model.x[j] >= r[j]
model.contract_constraint = Constraint(P, rule=fixed_contract_rule)

model.dual = Suffix(direction=Suffix.IMPORT)
results = opt.solve(model)

model.x.get_values()
```

Out[2]:
```
{'Microscope': 40.0,
 'Thermometer': 103.757251854717,
 'Safety Goggles': 114.16007407147,
 'Bunsen Burner': 40.8016595759643}
```

→ for part (a): ≈ 104 thermometers are produced

In [3]:
```
model.profit.expr()
```

Out[3]:
```
3424.968648547291
```

In [4]:
```
for i in T:
    print ('{0:14s}: {1:6.2f}, {2:6.2f}'.format(i,
           model.dual[model.capacity_constraint[i]],
           model.capacity_constraint[i].uslack()))
```

```
Team 1        :  12.00,  -0.00
Team 2        :   3.25,  -0.00
Team 3        :  26.07,  -0.00
```

shadow prices for questions (b) and (c)

In [5]:
```
for j in P:
    print ('{0:14s}: {1:6.2f}, {2:6.2f}'.format(j,
           model.dual[model.contract_constraint[j]],
           model.contract_constraint[j].lslack()))
```

```
Microscope    :  -1.40,   0.00
Thermometer   :   0.00, 103.76
Safety Goggles:   0.00, 114.16
Bunsen Burner :   0.00,  20.80
```

shadow price for the minimum supply constraint of the microscope for part (d)

3. Suppose each of the following tableaus occurs in the course of performing the simplex algorithm on a linear programming problem in standard form.

(a)

x_1	x_2	x_3	x_4	
0	-1	1	-1	2
1	0	0	2	3
0	-1	0	3	4

(a) Only negative entries in column x_2, so x_2 can be chosen arbitrarily large, making the problem feasible but unbounded.

(b)

x_1	x_2	x_3	x_4	
0	0	0	1	-1
0	0	1	0	1
2	1	0	0	10

(b) unfeasible (no other choice of basic variables is possible!)

new pivot

(c)

x_1	x_2	x_3	x_4	
2	1	0	1	1
1	0	1	4	3
-3	0	0	-1	8

State, for each case:

- Whether the problem is feasible.
- If the problem is feasible, whether the solution represented by the tableau is optimal.
- If the solution represented by the tableau is not optimal, if bounded optimal solutions exist and how to to choose the entering and leaving variable for the next iteration of the algorithm.

(5+5+5)

(c) feasible, not optimal.

Entering variable is x_1 leaving variable is x_2.

4. You are traveling home for the holidays. You have bought a number of presents for family and friends, listed in the table below. In your baggage, you have an allowance of 5 kg left. Which presents should you take along and which should you leave for another time if you intend to maximize the total value taken home?

Item	1	2	3	4	5
Value [EUR]	20 ~~15~~	5	25	10	20
Weight [kg]	2	1	4	1	3

Use dynamic programming to solve this problem. (No credit will be given for a brute-force solution which is possible for the small problem at hand, but does not scale well with problem size.)

(15)

Let stage $i=1,\dots,5$ correspond to the decision to pack item i.

Let s denote the remaining weight allowances.

Let $f_i(s,x_i)$ denote the value taken in the baggage at stage $\geq i$ with decision x_i.

Stage 5:

s	f_5^*	x_5^*	
0	0	0	(= leave)
1	0	0	
2	0	0	
3	20	1	(= take)
4	20	1	
5	20	1	

Stage 4:

s	$f_4(s,0)$	$f_4(s,1)$	f_4^*	x_4^*
0	0+0	—	0	0
1	0+0	10+0	10	1
2	0+0	10+0	10	1
3	0+20	10+0	20	0
4	0+20	10+20	30	1
5	0+20	10+20	30	1

Stage 3:

s	$f_3(s,0)$	$f_3(s,1)$	f_3^*	x_3^*
0	0+0	–	0	0
1	0+10	–	10	0
2	0+10	–	10	0
3	0+20	–	20	0
4	0+30	25+0	30	0
5	0+30	25+10	35	1

Stage 2:

s	$f_2(s,0)$	$f_2(s,1)$	f_2^*	x_2^*
0	0+0	–	0	0
1	0+10	5+0	10	0
2	0+10	5+10	15	1
3	0+20	5+10	20	0
4	0+30	5+20	30	0
5	0+35	5+30	35	0 or 1

Stage 1:

s	$f_1(s,0)$	$f_1(s,1)$	f_1^*	x_1^*
5	0+35	20+20	40	1

You should take items 1 and 5 only for a total value of 40 € and total weight of 5 kg.

5. Sailco manufactures sailboats. During the next 4 months the company must meet the following demands for their sailboats:

Month	1	2	3	4
Number of boats	40	60	70	25

At the beginning of Month 1, Sailco has 10 boats in inventory. Each month it must determine how many boats to produce. During any month, Sailco can produce up to 40 boats with regular labor and an unlimited number of boats with overtime labor. Boats produced with regular labor cost €400 each to produce, while boats produced with overtime labor cost €450 each. It costs €20 to hold a boat in inventory from one month to the next. The goal is to determine the production and inventory schedule that minimizes the cost of meeting the next 4 months' demands.

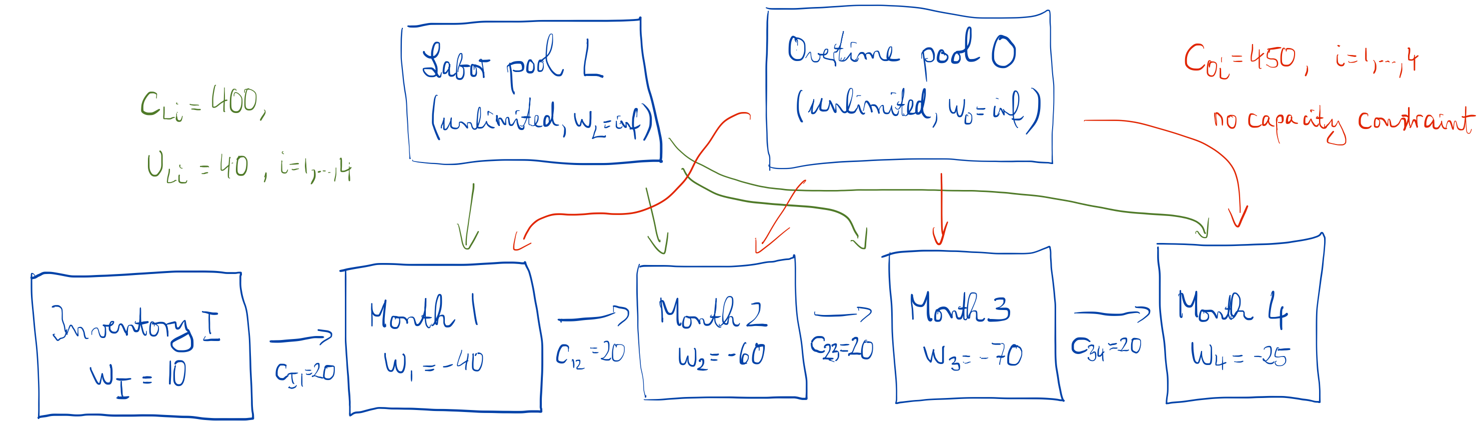
Since there are no setup costs for production, this problem can be formulated as a min cost flow problem.

(a) Show that this is indeed a min cost flow problem by drawing the network. Label all nodes and arcs with the relevant quantities from the text.

(b) Write out the corresponding linear program. You may use indexed variables for data and decision variables. There is no need to copy the concrete problem data from part (a). You do not need to solve this problem.

(5+5)

(a) Let the arcs denote boats or, equivalently, labor in units of boats.



(b) minimize $\displaystyle\sum_{i,j\in N} c_{ij} x_{ij}$

subject to $x_{ij} \le u_{ij}$, $i,j \in N$

$$\sum_{(i,j)\in A} x_{ij} - \sum_{(j,i)\in A} x_{ji} = w_j \qquad j \in N$$

$$x_{ij} \geq 0$$

6. You are about to buy a used car and you have narrowed your choice down to two possible models: one car is a private sale and the other is from a dealer. The cars are similar, and the only criterion is to minimize expected cost. The dealer car is more expensive, but it comes with a limited warranty. You decide that, if the car will last for 1 year, you can sell it again and recover a large part of your investment. If it falls apart, it will not be worth fixing. After test driving both cars and checking for obvious flaws, you make the following evaluation of probable resale values:

Car	**Purchase price**	**Est. resale value**	**Probability of lasting one year**
Private (A)	€800	€600	0.3
Dealer (B)	€1 500	€1 000	0.9

(a) Which car would you buy, based on risk-neutral valuation? (Assume that no replacement is required should the car break down.)

(b) What is the expected value of perfect information on car A?

(c) Suppose you have the opportunity to take car A to an independent mechanic who will charge €50 for doing an inspection and offer an opinion as to whether the car will last one year. Suppose you estimate the following probabilities to the accuracy of the mechanic's opinion:

	Mechanic says YES	**Mechanic says NO**
Car will last 1 year	0.7	0.3
Car won't last 1 year	0.1	0.9

Formulate this problem as a decision tree and determine the optimal strategy.

(5+5+10)

(a) expected cost of car A: $\frac{3}{10} \cdot 200 + \frac{7}{10} 800 = 60 + 560 = 620$

expected cost of car B: $\frac{9}{10} \cdot 500 + \frac{1}{10} \cdot 1500 = 450 + 150 = 600$

$\Rightarrow$ car B is the better choice.

(b) If perfect information asserts that car A will last, buy car A for a cost of $800 - 600 = 200$. Otherwise buy car B.

Total expected cost: $\frac{3}{10} \cdot 200 + \frac{7}{10} \cdot \underset{\text{from (a)}}{600} = 60 + 420 = 480$

$\Rightarrow$ EPVI for car A is $600 - 480 = 120$ €

(c) Let L be the event that car A will last

M be the event that the mechanic says it will last

Given: $P(M \mid L) = \frac{7}{10}$ $\quad P(\overline{M} \mid L) = \frac{3}{10}$

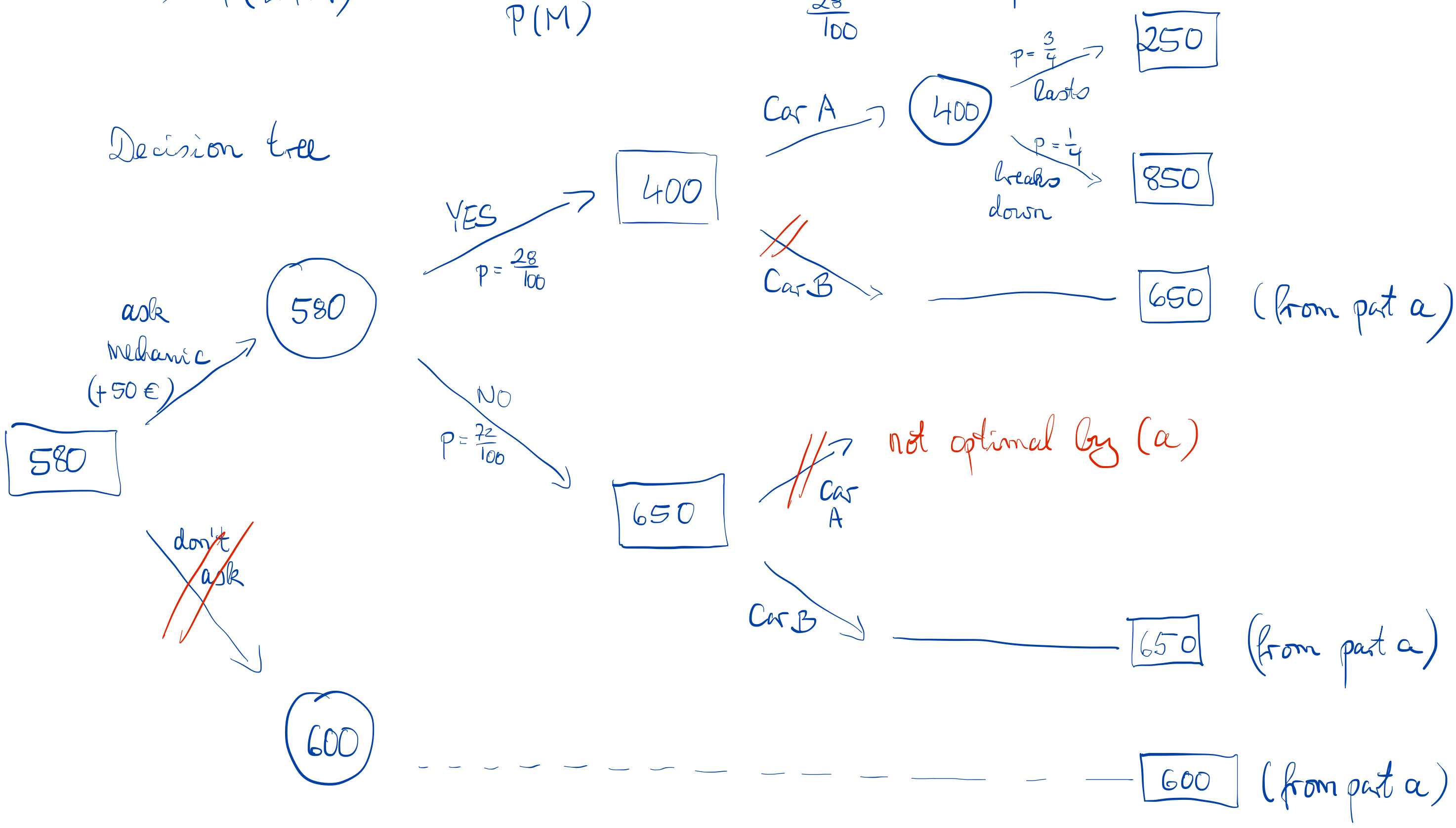
$P(M \mid \overline{L}) = \frac{1}{10}$ $\quad P(\overline{M} \mid \overline{L}) = \frac{9}{10}$

$$\Rightarrow P(M) = P(M \mid L)\,P(L) + P(M \mid \overline{L})\,P(\overline{L})$$

$$= \frac{7}{10}\,\frac{3}{10} + \frac{1}{10}\,\frac{7}{10} = \frac{28}{100}$$

$$\Rightarrow P(L \mid M) = \frac{P(M \mid L)\,P(L)}{P(M)} = \frac{\frac{7}{10} \cdot \frac{3}{10}}{\frac{28}{100}} = \frac{3}{4}$$

Decision tree



Best strategy: show car A to the mechanic, buy if mechanic says YES, otherwise buy car B, for a total expected cost of 580 €.