

# Operations Research

## Final Exam

### Instructions:

- Do all the work on this exam paper.
- Show your work, i.e., carefully write down the steps of your solution. You will receive points not just based on your final answer, but on the correct steps in your solution.
- No tools or other resources are allowed for this exam. In particular, no notes and no calculators.

**Name:** \_\_\_\_\_

**Matric. No.:** \_\_\_\_\_



**Problem 1: Graphical Method [25 points]**

Consider the following Linear Programming problem: Maximize

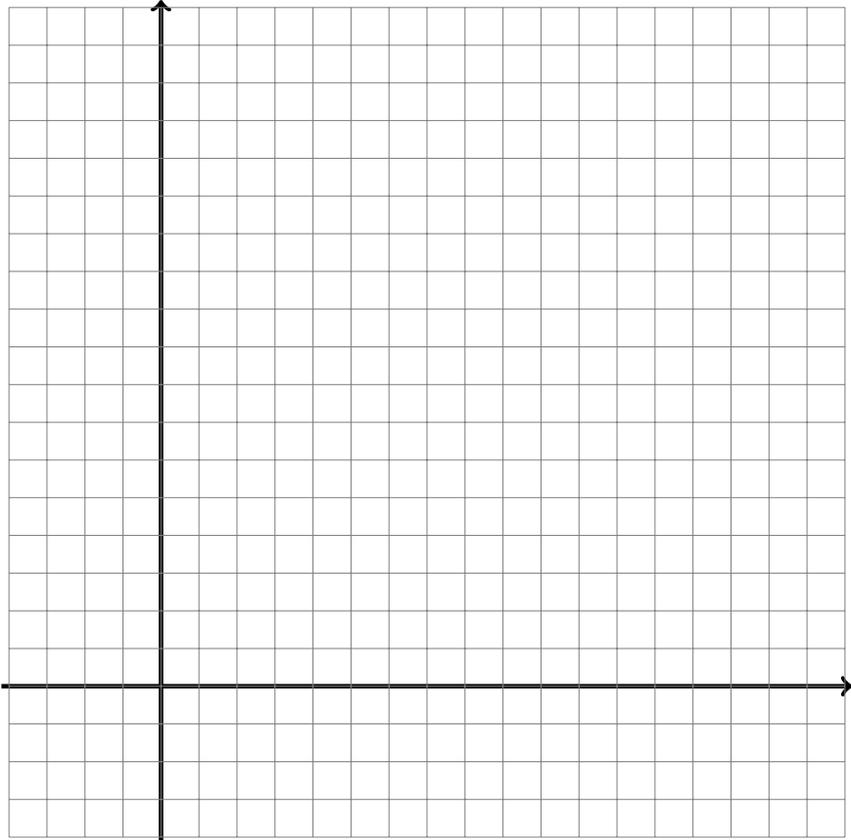
$$Z = x_1 + 3x_2$$

subject to

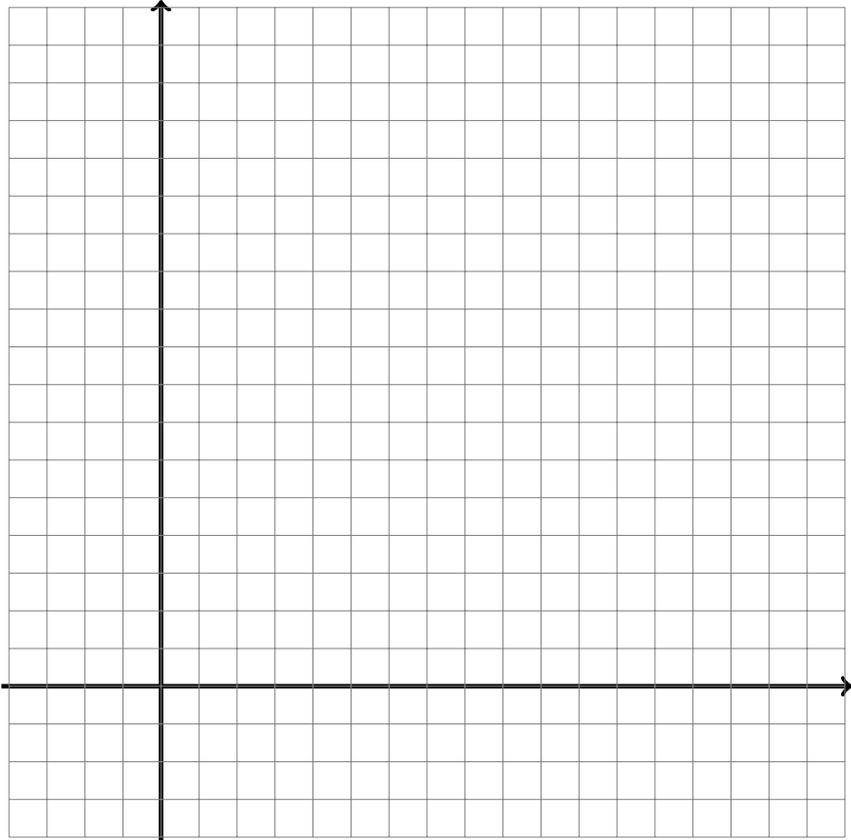
$$\begin{aligned} -x_1 + 2x_2 &\leq 6, \\ 3x_1 + 4x_2 &\leq 36, \\ x_2 &\geq 3, \\ x_1, x_2 &\geq 0. \end{aligned}$$

- (a) Solve the problem with the graphical method, i.e., draw the feasible region, and find the optimal solution and the corresponding value of  $Z$  graphically.
- (b) From part (a), identify the two binding constraints. Using this information, compute the optimal solution and the corresponding value of  $Z$ .
- (c) How many corner points does the feasible region have?
- (d) Write down the dual LP problem to the primal LP problem above. Without solving the dual problem explicitly, what is the value of  $b^T y$ , where  $b^T = (6, 36, -3)$ , and  $y$  is the optimal solution of the dual problem?
- (e) Find out which one of the following is the vector of shadow prices for this problem. Explain your answer! (Hint: You can figure out the answer directly from parts (b) and (d).)
- $(y_1, y_2, y_3) = (\frac{1}{2}, 0, \frac{1}{2})$ ,
  - $(y_1, y_2, y_3) = (0, \frac{1}{2}, \frac{1}{2})$ ,
  - $(y_1, y_2, y_3) = (\frac{1}{2}, \frac{1}{2}, 0)$ ,
  - $(y_1, y_2, y_3) = (0, \frac{1}{2}, \frac{3}{2})$ ,
  - $(y_1, y_2, y_3) = (\frac{3}{2}, \frac{1}{2}, 0)$ .

**Problem 1: Extra Space**



**Problem 1: Extra Space**



**Problem 1: Extra Space**

**Problem 2: Standard Form and Simplex Method [25 points]**

(a) Consider the following Linear Programming problem: Maximize

$$Z = x_1 + 2x_2$$

subject to

$$3x_1 + x_2 \leq 21,$$

$$x_1 + 3x_2 \leq 15,$$

$$x_1, x_2 \geq 0,$$

Write this problem in standard form, i.e., as the problem to minimize

$$\tilde{Z} = c^T \tilde{x}$$

subject to

$$A\tilde{x} = b, \quad \tilde{x} \geq 0.$$

Specify exactly the vectors  $b, c$  and the matrix  $A$ .

(b) Solve the problem with the simplex method as shown in class. (*Note: You will not receive points for simply stating the solution, but only for using the simplex method step by step.*)

**Problem 2: Extra Space**

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**Problem 3: Inventory Theory [25 points]**

We consider the following periodic review problem (which we also discussed in class):

- There are four production/delivery periods  $i = 1, 2, 3, 4$ .
- The demands in the four periods are  $r_1 = 3$ ,  $r_2 = 2$ ,  $r_3 = 3$ ,  $r_4 = 2$ , and they have to be met exactly.
- The setup costs for starting a production is  $K = 2$  (million Dollars).
- The holding costs for each unit left in inventory at the end of the period is  $h = 0.2$  (million Dollars).

The goal is to find the production schedule that minimizes the costs.

Note that the production cost per item is neglected here because it is the same in any production schedule. Also note that production should only be started if the inventory is empty (otherwise the production schedule is not optimal).

Find the optimal production schedule and total production cost *using dynamic programming*.

(Hint: Use the state  $s$  = number of items in inventory, and the decision variables  $x_i$  = number of items produced in period  $i$ . Then set up the correct cost function  $f_i(s, x_i)$ .)

**Problem 3: Extra Space**

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**Problem 4: Decision Analysis [25 points]**

Consider the decision analysis problem with the following payoff table.

Alternative	Payoff in State $S_i$	
	$S_1$	$S_2$
$A_1$	700	-100
$A_2$	90	90
Prior Probability	1/4	3/4

- (a) Compute the expected payoff (with the prior probabilities) for each alternative. Given the result, which alternative should be chosen?
- (b) Next, some experimentation is done in order to improve the decision making. The outcome of the experimentation can be positive (“pos”) or negative (“neg”). The probabilities that the outcome is positive/negative given state  $S_1/S_2$  are given by

$$P(\text{pos} | S_1) = 0.6,$$

$$P(\text{pos} | S_2) = 0.2,$$

$$P(\text{neg} | S_1) = 0.4,$$

$$P(\text{neg} | S_2) = 0.8.$$

- (i) Compute the total probabilities  $P(\text{pos})$  and  $P(\text{neg})$  for the outcome to be positive or negative. (*Note: The result for negative is  $P(\text{neg}) = 0.7$ ; points for this exercise will be given for the correct computation.*)
- (ii) Use Bayes’ rule to compute the probabilities that state  $S_1$  is realized, given a positive/negative outcome of the experimentation, i.e., compute  $P(S_1 | \text{pos})$  and  $P(S_1 | \text{neg})$ . (*Note: The results are  $P(S_1 | \text{pos}) = 0.5$  and  $P(S_1 | \text{neg}) = 1/7$ ; points for this exercise will be given for the correct computation.*)
- (c) Given the results of part (b), set up the decision tree for this problem. The expected payoffs for each step do *not* need to be computed! The tree needs to include the possibilities “with/without experimentation”, “positive/negative outcome”, “alternative  $A_1/A_2$ ”, “state  $S_1/S_2$ ”. The tree should include a variable  $c$  for the cost of experimentation, the necessary probabilities, and the total payoffs at the end only (no intermediate payoffs need to be computed).

**Problem 4: Extra Space**

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**Problem 5: Pyomo [25 points]**

The pyomo program on the last exam page shows an implementation of a minimum cost flow problem.

- (a) Draw the corresponding network representation of the problem. (Make sure to include the given values of the parameters from the pyomo program.)
- (b) Write down the Linear Programming problem that is solved in mathematical notation (i.e., write down the objective function and the constraints).
- (c) Briefly explain the meaning of the parameters  $b, C, U$  in the context of minimum cost flow problems. In particular, explain what positive, negative, or zero values for entries in  $b$  mean.
- (d) Suppose one of the entries in  $U$  can be slightly increased. Which one should we choose in order to lower the costs? (You will receive points only if you justify your answer correctly, and not for guessing.)
- (e) Suppose one of the entries F1, F2, or F3 in  $b$  is increased by one unit, and simultaneously one of the entries W1, W2, or W3 in  $b$  is decreased by one unit. Which entries should be increased/decreased in order to be most cost effective? What will be the extra cost for the increase/decrease?
- (f) Management notices that there is some flexibility between F1, F2, and F3 concerning the units of  $b$ , i.e., one of them could be increased by a few units while simultaneously decreasing another one by the same number of units. Which of the following four solutions should we recommend in order to be most cost effective:
  - 1. Increase F1 by a few units and decrease F2.
  - 2. Increase F2 by a few units and decrease F3.
  - 3. Increase F3 by a few units and decrease F1.
  - 4. Increase F3 by a few units and decrease F2.

Explain your answer. (You will not receive points for a random choice.)

**Problem 5: Extra Space**

**Problem 5: Extra Space**

**Problem 5: Extra Space**

```
In [1]: from pyomo.environ import *
        from pyomo.opt import *
        opt = solvers.SolverFactory("glpk")
```

```
In [2]: b = {'F1':20,
            'F2':50,
            'F3':30,
            'DC1':0,
            'DC2':0,
            'W1':-30,
            'W2':-20,
            'W3':-50}

        C = (('F1', 'DC1'):200,
            ('F2', 'DC1'):100,
            ('F2', 'DC2'):300,
            ('F3', 'DC2'):200,
            ('DC1', 'W1'):400,
            ('DC1', 'W2'):500,
            ('DC2', 'W2'):400,
            ('DC2', 'W3'):600}

        U = (('F2', 'DC1'):40,
            ('DC1', 'W2'):10}

        N = list(b.keys())
        A = list(C.keys())
        V = list(U.keys())
```

```
In [3]: model = ConcreteModel()
        model.x = Var(A, within=NonNegativeReals)
```

```
In [4]: def flow_rule(model, n):
        InFlow = sum(model.x[i,j] for (i,j) in A if j==n)
        OutFlow = sum(model.x[i,j] for (i,j) in A if i==n)
        return OutFlow - InFlow == b[n]

        model.flow = Constraint(N, rule=flow_rule)
```

```
In [5]: def capacity_rule(model, i, j):
        return model.x[i,j] <= U[i,j]

        model.capacity = Constraint(V, rule=capacity_rule)
```

```
In [6]: model.cost = Objective(expr = sum(model.x[a]*C[a] for a in A), sense=minimize)
```

```
In [7]: model.dual = Suffix(direction=Suffix.IMPORT)
        results = opt.solve(model)
        model.x.get_values()
```

```
Out[7]: {'F1', 'DC1'): 20.0,
         ('F2', 'DC1'): 20.0,
         ('F2', 'DC2'): 30.0,
         ('F3', 'DC2'): 30.0,
         ('DC1', 'W1'): 30.0,
         ('DC1', 'W2'): 10.0,
         ('DC2', 'W2'): 10.0,
         ('DC2', 'W3'): 50.0}
```

```
In [8]: model.cost.expr()
```

```
Out[8]: 72000.0
```

```
In [9]: for n in N:
        print(n, model.dual[model.flow[n]])
```

```
F1 100.0
F2 0.0
F3 -100.0
DC1 -100.0
DC2 -300.0
W1 -500.0
W2 -700.0
W3 -900.0
```

```
In [10]: for j in V:
        print(j, model.dual[model.capacity[j]])
```

```
('F2', 'DC1') 0.0
('DC1', 'W2') -100.0
```

