Operations Research, Jacobs University, Fall 2022 Session 4

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Goal: Write LP problems in the following **standard form**:

- Minimize $z = c^T x$, with $c \in \mathbb{R}^m$, $x \in \mathbb{R}^m$
- Constraints: $Ax = b$, with A an $n \times m$ matrix, $b \in \mathbb{R}^n$
  and $x \geq 0$ (meaning $x_j \geq 0$ for all $j = 1, \ldots, m$)

Note: minimization is just a convention here.

We illustrate this with the following example ("proof by example"):

- Maximize $z = x_1 + 2x_2 + 3x_3$
- Constraints:

$$x_1 + x_2 - x_3 = 1 \qquad (1)$$

$$-2x_1 + x_2 + 2x_3 \geq -5 \qquad (2)$$

$$x_1 - x_2 \leq 4 \qquad (3)$$

$$x_2 + x_3 \leq 5 \qquad (4)$$

$$x_1 \geq 0 \qquad (5)$$

$$x_2 \geq 0 \qquad (6)$$

Step 1: Turn maximization into minimization.

Our example: Minimize $\tilde{z} = -x_1 - 2x_2 - 3x_3$

Step 2: Slack variables.

↳ First, write inequalities in standard order: all variables to the left, number to the right, $\leq$ sign.

Our example: Write ③ as $2x_1 - x_2 - 2x_3 \leq 5$.

↳ Then, turn inequalities into equalities + non-negativity constraints by introducing "slack variables" s_i:

Our example: Write ②, ③, ④ as:

$$2x_1 - x_2 - 2x_3 + s_1 = 5 \quad \text{with} \quad s_1 \geq 0,$$

$$x_1 - x_2 + s_2 = 4 \quad \text{with} \quad s_2 \geq 0,$$

$$x_2 + x_3 + s_3 = 5 \quad \text{with} \quad s_3 \geq 0.$$

Step 3: Replace variables without non-negativity constraint by differences.

Our example: x_3 has no nonnegativity constraint

$\Rightarrow$ write $x_3 = u - v$ with $u \geq 0, v \geq 0$.

To summarize, we have rewritten the problem in standard form with:

$$\tilde{x} = \begin{pmatrix} x_1 \\ x_2 \\ u \\ v \\ s_1 \\ s_2 \\ s_3 \end{pmatrix}, \quad A = \begin{array}{c} \begin{array}{ccccccc} x_1 & x_2 & u & v & s_1 & s_2 & s_3 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \end{array} \\ \begin{pmatrix} 1 & 1 & -1 & 1 & 0 & 0 & 0 \\ 2 & -1 & -2 & 2 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 0 & 1 \end{pmatrix} \end{array}, \quad b = \begin{pmatrix} 1 \\ 5 \\ 4 \\ 5 \end{pmatrix}, \quad c = \begin{pmatrix} -1 \\ -2 \\ -3 \\ 3 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$(m = 7, n = 4)$

$(\tilde{z} = c^T \tilde{x}, \ A\tilde{x} = b, \ \tilde{x} \geq 0)$.

We are now confronted with solving a system of linear eq.s $Ax = b$, with $A \in \mathrm{Mat}(n \times m)$, $b \in \mathbb{R}^n$.

Note:

- As in the example above, for us A is typically a wide matrix $(m > n)$, i.e., the system is underdetermined and there are many solutions.
- In Finite Mathematics you learned about least-norm solutions, i.e., solutions that minimize $\|x\|$. Our goal is: Find solution that optimizes the linear objective function.

First: How do we find all solutions to $Ax = b$? (Afterwards: How do we select the optimal one?)

↳ Use Gaussian elimination to bring augmented matrix into $\boxed{\text{reduced row echelon form}}$.

- All rows with just 0's at the bottom
- leading coefficient of a nonzero row is to the right of the one from row above
- All leading coefficients are 1 (they are called "pivots")
- Each column with a leading 1 has 0's in all other entries

E.g.:
$$\begin{pmatrix} 1 & 0 & 5 & 0 & 8 \\ 0 & 1 & 3 & 0 & 4 \\ 0 & 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$
pivots

Ex.: $A = \begin{pmatrix} 1 & 3 & 1 & 1 \\ 2 & 6 & 0 & -1 \end{pmatrix}$, $b = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\Rightarrow$ augmented matrix: $\left(\begin{array}{cccc|c} 1 & 3 & 1 & 1 & 2 \\ 2 & 6 & 0 & -1 & 1 \end{array}\right)$

reduced row-echelon form:

$-2R_1 + R_2 \to R_2$: $\left(\begin{array}{cccc|c} 1 & 3 & 1 & 1 & 2 \\ 0 & 0 & -2 & -3 & -3 \end{array}\right)$

$R_2/-2$: $\left(\begin{array}{cccc|c} 1 & 3 & 1 & 1 & 2 \\ 0 & 0 & 1 & \frac{3}{2} & \frac{3}{2} \end{array}\right)$

$R_1 - R_2 \to R_1$: $\left(\begin{array}{cccc|c} 1 & 3 & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{3}{2} & \frac{3}{2} \end{array}\right)$

pivots

This corresponds to the two equations

$$x_1 + 3x_2 \qquad - \tfrac{1}{2} x_4 = \tfrac{1}{2}$$
$$x_3 + \tfrac{3}{2} x_4 = \tfrac{3}{2}$$

Solve these two equations: next time