Operations Research, Jacobs University, Fall 2022 — Session 10

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2.4 The Dual LP Problem

Let us consider another example:

A factory produces cars (x_1) and trucks (x_2). We aim at maximizing the profit $z = 3x_1 + 2x_2$, subject to the constraints

profit per car (3), profit per truck (2) → minimize $z' = -3x_1 - 2x_2$

$5x_1 \le 100$ (car assembly; need 5 h per car; 100 h available) (1)

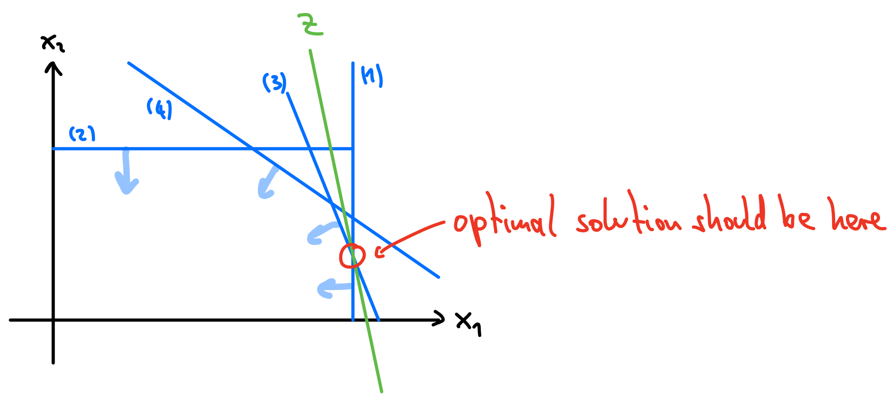
$10x_2 \le 100$ (truck assembly) (2)

$4x_1 + 3x_2 \le 100$ (metal stamping) (3)

$3x_1 + 5x_2 \le 100$ (engine assembly) (4)

$x_1, x_2 \ge 0$

Graphically (rough sketch):



Now we would compute the solution via the simplex method:

x_1	x_2	s_1	s_2	s_3	s_4	
5	0	1	0	0	0	100
0	10	0	1	0	0	100
4	3	0	0	1	0	100
3	5	0	0	0	1	100
-3	-2	0	0	0	0	0

entry variable: x_1

leaving variable: s_1

	x_1	x_2	s_1	s_2	s_3	s_4	
$R_1/5 \to R_1$:	1	0	$\frac{1}{5}$	0	0	0	20
	0	10	0	1	0	0	100
$-\frac{4}{5}R_1 + R_3 \to R_3$:	0	3	$-\frac{4}{5}$	0	1	0	20
$-\frac{3}{5}R_1 + R_4 \to R_4$:	0	5	$-\frac{3}{5}$	0	0	1	40
$\frac{3}{5}R_1 + R_5 \to R_5$:	0	-2	$\frac{2}{5}$	0	0	0	60

entry variable: x_2

leaving variable: s_3

	x_1	x_2	s_1	s_2	s_3	s_4	
	1	0	$\frac{1}{5}$	0	0	0	20
$-\frac{10}{3}R_3 + R_2 \to R_2$:	0	0	$\frac{40}{15}$	1	$-\frac{10}{3}$	0	$\frac{100}{3}$
$R_3/3 \to R_3$:	0	1	$-\frac{4}{15}$	0	$\frac{1}{3}$	0	$\frac{20}{3}$
$-\frac{5}{3}R_3 + R_4 \to R_4$:	0	0	$\frac{11}{15}$	0	$-\frac{5}{3}$	1	$\frac{20}{3}$
$\frac{2}{3}R_3 + R_5 \to R_5$:	0	0	$\frac{1}{15}$	0	$\frac{2}{3}$	0	$\frac{220}{3}$

Done!

$\Rightarrow$ Optimal solution: $x_1 = 20$, $x_2 = \frac{20}{3}$, $s_2 = \frac{100}{3}$, $s_4 = \frac{20}{3}$, $s_1 = 0$, $s_3 = 0$, with $z' = -\frac{220}{3}$

(meaning profit $z = -z' = \frac{220}{3}$)

Now note:

- Constraints (1) and (3) hold with equality (no slack: $s_1 = 0, s_3 = 0$). These are the binding constraints.
- Constraints (2) and (4) are non-binding.

Generally:

- We have n physical variables, m slack variables (constraints); here: $n=2, m=4$.
- Unless there is redundancy, m variables are basic.
- Typically, all physical variables are basic, so also $m-n$ slack variables are basic.
- Thus, n slack variables are typically non-basic ($=0$ in optimal solution): These correspond to the binding constraints.

Knowing our optimal solution, we could forget about the non-binding constraints, i.e., we can write our solution in the following way:

(1) and (3) are the binding constraints

$$\text{Let } \tilde{A} = \begin{array}{cc} x_1 & x_2 \\ \end{array}\begin{array}{l}\left(\begin{array}{cc} 5 & 0 \\ 4 & 3 \end{array}\right)\begin{array}{l} \leftarrow \text{constraint (1)} \\ \leftarrow \text{constraint (3)} \end{array}\end{array} \qquad \tilde{b} = \begin{pmatrix} 100 \\ 100 \end{pmatrix}, \text{ want } \tilde{A}x = \tilde{b}.$$

Since $\tilde{A}$ is now a square $(n \times n)$ matrix with full rank, it can be inverted: $x = \tilde{A}^{-1}\tilde{b}$.

$$\Rightarrow z' = \underbrace{c^T}_{\substack{=(-3,-2) \\ \text{in our example}}} x = \underbrace{c^T \tilde{A}^{-1}}_{=:\tilde{y}^T} \tilde{b} = y^T b$$

, where $\tilde{y} \in \mathbb{R}^2$, $y \in \mathbb{R}^4$ filled up with zeros.

i.e., here $y = \begin{pmatrix} \vdots \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

Note: For 2×2 matrices, the formula for the inverse is: $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

Here concretely: $\tilde{y}^T = c^T \tilde{A}^{-1} = (3,2)\begin{pmatrix} 5 & 0 \\ 4 & 3 \end{pmatrix}^{-1} = (3,2)\,\frac{1}{15}\begin{pmatrix} 3 & 0 \\ -4 & 5 \end{pmatrix}$

$= \frac{1}{15}(1, 10)$. $\quad \left(\Rightarrow y^T = \frac{1}{15}(1, 0, 10, 0)\right)$

Test: $\tilde{y}^T \tilde{b} = \frac{1}{15}(1, 10)\begin{pmatrix} 100 \\ 100 \end{pmatrix} = \frac{1100}{15} = \frac{220}{3} \quad (= z)$ ✓