Operations Research, Jacobs University, Fall 2022 — Session 19

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We continue the example from last time

# of medical teams	Countries 1	2	3
0	0	0	0
1	45	20	50
2	70	45	70
3	90	75	80
4	105	110	100
5	120	150	130

← # of additional person-years of life saved (in thousands)

(note: not proportional to # of teams!)

Goal: distribute 5 medical teams to the three countries s.t. maximum person-years saved.

Note: This problem type is called "distribution of effort problem".

Dynamic Programming formulation:

- stages = countries, x_i = # of teams sent to country i, s = # of teams still available
- here: $f(s,x_i) = \underbrace{b_{s,x_i}}_{\text{see table above}} + f^*_{i+1}(\underbrace{s-x_i}_{\text{\# of teams available in next step (the new } s \text{ in the next step)}})$

Solution:

$i=3$: (country 3)

s	f_3^*	x_3^*
0	0	0
1	50	1
2	70	2
3	80	3
4	100	4
5	130	5

$i=2$:

s	$x_2=0$	$x_2=1$	$x_2=2$	$x_2=3$	$x_4=4$	$x_2=5$	$f_2^*(s)$	x_2^*
	$f_2(s,x_2)=b_{s,x_2}+f_3^*(s-x_2)$							
0	0+0	–	–	–	–	–	0	0
1	0+50	20+0	–	–	–	–	50	0
2	0+70	20+50	45+0	–	–	–	70	0 or 1
3	0+80	20+70	45+50	75+0	–	–	95	2
4	0+100	20+80	45+70	75+50	110+0	–	125	3
5	0+130	20+100	45+80	75+70	110+50	150+0	160	4

$i=1$:

s here

s	$x_1=0$	$x_1=1$	$x_1=2$	$x_1=3$	$x_1=4$	$x_1=5$	$f_1^*(s)$	x_1^*
	$f_1(s,x_1)=b_{s,x_1}+f_2^*(5-x_1)$							
5	0+160 =160	45+125 =170	70+95 =165	90+70 =160	105+50 =155	120+0 =120	170	1

$\Rightarrow$ Solution: send 1 team to country 1 (4 remaining), 3 to country 2 (1 remaining), and 1 to country 3, to save the maximal 170 thousand person-years of life.

Example: Local job shop problem (Hillier, Lieberman: Chapter 11.3)

Setup: different seasons with minimal worker requirements:

Season	Spring	Summer	Autumn	Winter	Spring	...
Requirement	255	220	240	200	255	

Too much employment costs 2000\$ per person per season.

Changing employment from one season to the next costs 200\$ · (difference in employment)^2.

We assume fractional employment is possible (part-time work). (I.e., optimal solution need not be integer.)

Goal: Find hiring schedule that minimizes costs.

We introduce: • Stage 4 = spring, Stage 3 = winter, Stage 2 = autumn, Stage 1 = summer

• x_n = employment level for stage n. Note: $x_4 = 255$.

• r_n = minimum requirements from table above.

$\Rightarrow$ Feasible values for x_n: $r_n \leq x_n \leq 255$

Note: The states $s_n = x_{n-1}$ can now take a continuum of values.

Similar to before, we set

$$f_n(s_n, x_n) = \underbrace{200(x_n - s_n)^2}_{\text{employment change cost}} + \underbrace{2000(x_n - r_n)}_{\text{extra employment cost}} + \underbrace{f^*_{n+1}(x_n)}_{\text{optimal costs for later stages}}$$

} cost given employment s_n at stage $n-1$, and x_n at stage n, and optimal future cost

$$f^*_n(s_n) = \min_{r_n \leq x_n \leq 255} f_n(s_n, x_n)$$

} optimal cost given employment s_n at stage $n-1$, and optimal in the future

Solution:

Stage $n = 4$:

s_4	$f^*_4(s_4)$	x^*_4
$200 \leq s_4 \leq 255$	$200(255 - s_4)^2$	255

Stage $n = 3$: $f^*_3(s_3) = \min_{200 \leq x_3 \leq 255} f_3(s_3, x_3)$

$$= \min_{200 \leq x_3 \leq 255} \left[200(x_3 - s_3)^2 + 2000(x_3 - 200) + \underbrace{200(255 - x_3)^2}_{f^*_4(x_3)} \right]$$

We find the minimum by setting the partial derivative w.r.t. x_3 to zero (keeping s_3 fixed):

$$\frac{\partial f_3(s_3, x_3)}{\partial x_3} = 400(x_3 - s_3) + 2000 - 400(255 - x_3) = 400(2x_3 - s_3 - 250) \stackrel{!}{=} 0$$

$$\Rightarrow \quad x_3^* = \frac{250+s_3}{2}$$

Since $\dfrac{\partial^2 f_3(s_3,x_3)}{\partial x_3^2} = 800 > 0$, we indeed have a minimum.

Possible s_3: $240 \le s_3 \le 255 \Rightarrow 245 \le x_3^* \le 252.5$ is feasible $(200 \le x_3 \le 255)$, so x_3^* is always the feasible minimum.

$$\Rightarrow f_3^*(s_3) = 200(x_3^* - s_3)^2 + 2000(x_3^* - 200) + 200(255 - x_3^*)^2$$

$$= 200\underbrace{\left(\frac{250+s_3}{2} - s_3\right)^2}_{=\left(\frac{250-s_3}{2}\right)^2 = \frac{1}{4}(250-s_3)^2} + 2000\underbrace{\left(\frac{250+s_3}{2} - 200\right)}_{=\frac{-150+s_3}{2}} + 200\underbrace{\left(255 - \left(\frac{250+s_3}{2}\right)\right)^2}_{=\left(\frac{260-s_3}{2}\right)^2}$$

$\Rightarrow$

s_3	$f_3^*(s_3)$	x_3^*
$240 \le s_3 \le 255$	$50(250-s_3)^2 + 50(260-s_3)^2 + 1000(s_3-150)$	$\frac{250+s_3}{2}$

One proceeds similarly with Stages 2 and 1, but we skip the details.

The results are:

$n=2$:

s_2	$f_2^*(s_2)$	x_2^*
$220 \le s_2 \le 240$	$200(240-s_2)^2 + 115.000$	240
$240 \le s_2 \le 255$	$\frac{200}{9}\left[(240-s_2)^2 + (255-s_2)^2 + (270-s_2)^2\right] + 2000(s_2-195)$	$\frac{2s_2+240}{3}$

$n=1$:

s_1	$f_1^*(s_1)$	x_1^*
255	185.000	247.5

$\Rightarrow x_1^* = 247.5$, $x_2^* = 245$, $x_3^* = 247.5$, $x_4^* = 255$, and minimum cost is 185.000 \$.