Operations Research, Jacobs University, Fall 2022 — Session 23

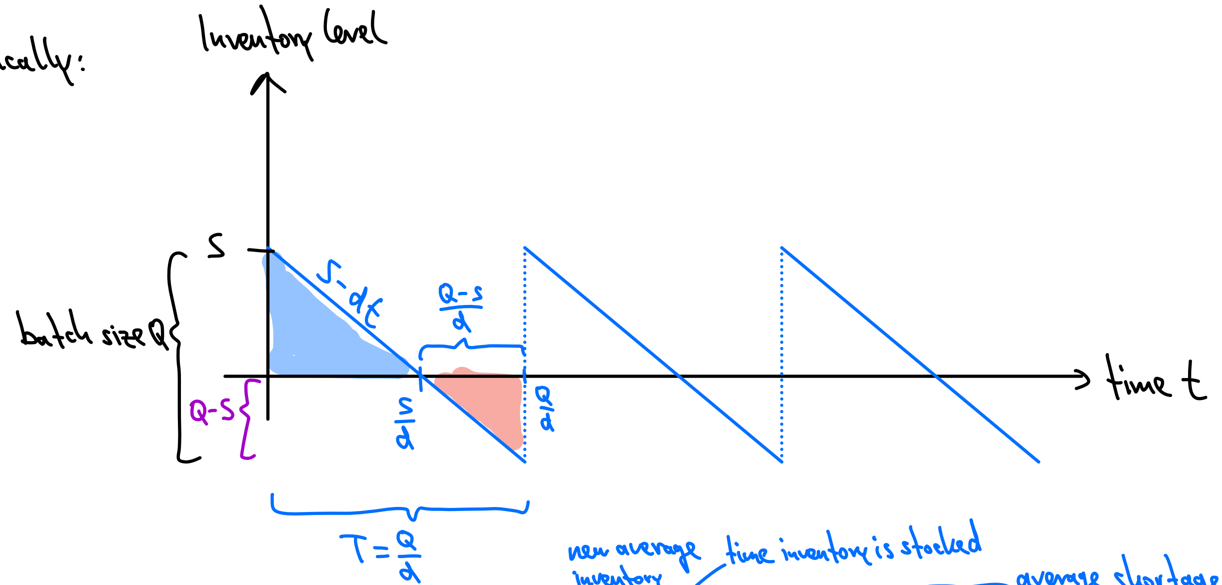
Prof. Sören Petrat — Nov. 22, 2022

We continue our discussion of inventory management.

Let us assume the inventory can be empty for part of a cycle at a penalty p per unit per time. Withdrawels that cannot be fulfilled will be postponed and processed when new batch arrives.

This leads to the EOQ model with planned shortages

Graphically:



$$\Rightarrow \text{Cycle cost } C_{cycle} = \underbrace{K + cQ}_{\text{order cost}} + \underbrace{\boxed{h \underbrace{\frac{S}{2}}_{\text{new average inventory}} \underbrace{\frac{S}{d}}_{\text{time inventory is stocked}}}}_{\text{holding cost}} + \underbrace{\boxed{p \underbrace{\frac{Q-S}{2}}_{\text{average shortage}} \underbrace{\frac{Q-S}{d}}_{\text{shortage time}}}}_{\text{penalty}}$$

$$\Rightarrow \text{Total cost per time } C = \frac{C_{cycle}}{T} = \frac{K}{Q/d} + \frac{cQ}{Q/d} + \frac{1}{2}\frac{hS^2}{d\,Q/d} + \frac{1}{2}p\frac{(Q-S)^2}{d\,Q/d}$$

$$= \frac{dK}{Q} + dc + \frac{1}{2}\frac{hS^2}{Q} + \frac{1}{2}p\underbrace{\frac{(Q-S)^2}{Q}}_{= \frac{(Q-S)}{Q}(Q-S) = \left(1-\frac{S}{Q}\right)(Q-S)}$$

Here, Q and S are decision variables, so we need to compute two partial derivatives:

$$\frac{\partial C}{\partial S} = \frac{hS}{Q} - p\,\frac{(Q-S)}{Q} \overset{!}{=} 0 \quad \Rightarrow \quad \underbrace{hS = p(Q-S)}_{(*)} \quad \Rightarrow (h+p)\,S = pQ$$

$$\Rightarrow S = \frac{p}{h+p}\,Q$$

$$\frac{\partial C}{\partial Q} = -\frac{dK}{Q^2} - \frac{1}{2}\frac{hS^2}{Q^2} + \frac{1}{2}p\left(\frac{S}{Q^2}(Q-S) + \underbrace{1-\frac{S}{Q}}_{=\frac{Q-S}{Q}}\right)$$

$$= -\frac{dK}{Q^2} - \frac{1}{2}\frac{hS^2}{Q^2} + \frac{1}{2}\underbrace{p(Q-S)}_{=hS \text{ (see Equation } (*))}\left(\frac{S}{Q^2} + \frac{1}{Q}\right) \overset{!}{=} 0 \qquad (*)$$

$$\Rightarrow (*) \Rightarrow \quad -\frac{dK}{Q^2} - \frac{1}{2}\frac{hS^2}{Q^2} + \frac{1}{2}hS\left(\frac{S}{Q^2} + \frac{1}{Q}\right) = 0$$

$$\Leftrightarrow \quad -\frac{dK}{Q^2} + \frac{1}{2}\frac{hS}{Q} = 0$$

$S = \frac{p}{h+p}Q$

$$\Rightarrow \quad \frac{dK}{Q^2} = \frac{1}{2}\frac{hp}{h+p}$$

$$\Rightarrow \quad \boxed{Q^* = \underbrace{\sqrt{\frac{2dK}{h}}}_{\text{known part from previous EOQ formula}}\sqrt{\frac{h+p}{p}}} \quad \text{is the minimum}$$

with corresponding $S^* = \frac{p}{h+p}Q^* = \sqrt{\frac{2dK}{h}}\sqrt{\frac{p}{h+p}}$,

and cycle time $T^* = \frac{Q^*}{d} = \sqrt{\frac{2K}{dh}}\sqrt{\frac{p}{h+p}}$.

Note: If $\underbrace{p \to \infty}_{\text{very high penalty, so no shortage should be optimal}}$, then $\sqrt{\frac{h+p}{p}} \to 1$, and we recover the basic EOQ model from before.