Stochastic Methods Lab, Jacobs University Bremen, Fall 2022 — Session 6, Sep. 22, 2022

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1.4 Immunization

Idea: Reduce risk from changes in interest rate if a future liability L has to be met at period m (here, m is called "horizon").

Simplest possibility: cash-flow matching: buy zero-coupon bond with maturity m and par value $F = L$.

But this has some practical disadvantages, e.g., bonds with needed maturity might not exist (+ potentially lower yields, see later).

Alternative idea: buy two bonds, one with maturity $n_1 < m$, and one with maturity $n_2 > m$.

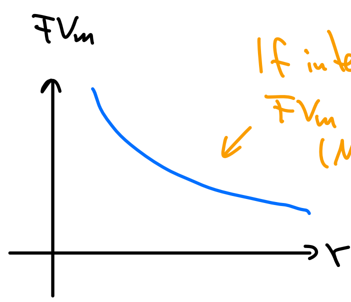
Take a zero-coupon bond: After m years, the future value FV_m of a bond with maturity n and price P, par value F is:

$$FV_m = (1+r)^m P$$

$$= (1+r)^m \frac{F}{(1+r)^n}$$

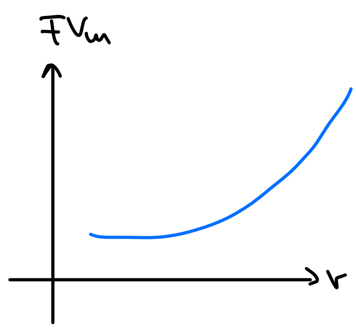
$$= F (1+r)^{m-n}$$

If $m < n$:



If interest rates rise, FV_m declines. (Not good here)

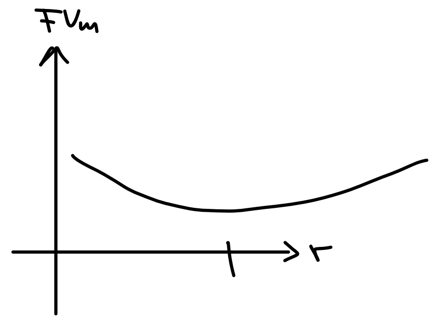
If $m > n$:



Now take 2 bonds: • F_1, $n_1 < m$

• F_2, $n_2 > m$

$\Rightarrow$ Future value of this portfolio is



$$FV_m = F_1(1+r)^{m-n_1} + F_2(1+r)^{m-n_2}$$

Now we have the following conditions:

(1) $FV_m \stackrel{!}{=} L$ to meet liability

(2) Stability w.r.t. changes in r

For (2): Find minimum of $FV_m(r)$

$$\Rightarrow \frac{\partial FV_m(r)}{\partial r} = (m-n_1)F_1(1+r)^{m-n_1-1} + (m-n_2)F_2(1+r)^{m-n_2-1} \stackrel{!}{=} 0$$

$$\Rightarrow (m-n_1)F_1(1+r)^{-n_1} + (m-n_2)F_2(1+r)^{-n_2} = 0$$

$$\Rightarrow m\underbrace{\frac{F_1}{(1+r)^{n_1}}}_{P_1} + m\underbrace{\frac{F_2}{(1+r)^{n_2}}}_{P_2} = n_1\underbrace{\frac{F_1}{(1+r)^{n_1}}}_{P_1} + n_2\underbrace{\frac{F_2}{(1+r)^{n_2}}}_{P_2}$$

With total price $P = P_1 + P_2$ we get the condition

$$\boxed{m = \underbrace{\frac{1}{P}\left(n_1P_1 + n_2P_2\right)}_{\text{weighted average price}} =: MD} \quad (\text{Macaulay duration})$$

Note: price volatility is defined as $V = -\frac{1}{P}\frac{\partial P}{\partial r}$

Here, $P = \frac{F_1}{(1+r)^{n_1}} + \frac{F_2}{(1+r)^{n_2}}$, so $\frac{\partial P}{\partial r} = -n_1(1+r)^{-1}\underbrace{\frac{F_1}{(1+r)^{n_1}}}_{P_1} - n_2(1+r)^{-1}\underbrace{\frac{F_2}{(1+r)^{n_2}}}_{P_2}$

$$= -(1+r)^{-1}\left(n_1 P_1 + n_2 P_2\right)$$

$$\Rightarrow \boxed{MD = (1+r)\left(-\frac{1}{P}\frac{\partial P}{\partial r}\right)}$$

$\Rightarrow$ Conclusion: set up portfolio with $MD = m$.

Note: Are we really at a minimum?

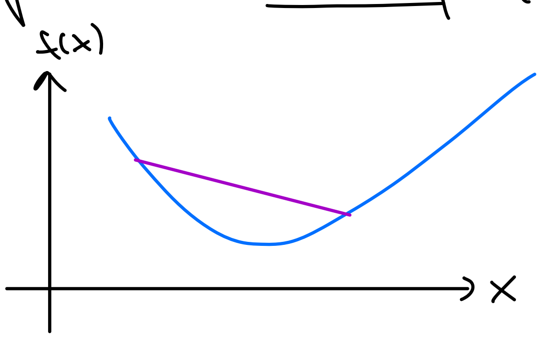
Check $\frac{\partial^2 FV_m(r)}{\partial r^2} = \frac{\partial}{\partial r}\left[(m-n_1)F_1(1+r)^{m-n_1-1} + (m-n_2)F_2(1+r)^{m-n_2-1}\right]$

$$= \underbrace{(m-n_1)}_{>0}\underbrace{(m-n_1-1)}_{\geq 0}F_1(1+r)^{m-n_1-2} + \underbrace{(m-n_2)}_{<0}\underbrace{(m-n_2-1)}_{<0}F_2(1+r)^{m-n_2-2}$$

$$> 0$$

$\Rightarrow$ Minimum

More general: Need <u>convexity</u> for minimum



$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda)f(x_2)$$

$$\forall \lambda \in [0,1]$$

$\Rightarrow$ 3 Conditions for immunization:

(1) $FV_m \overset{!}{=} L$ to meet liability.

(2) $m = MD$.

(3) $FV_m(r)$ convex around relevant r.

Note:

- For a general cash flow $P = \sum_{i=1}^{n} \frac{C_i}{(1+r)^i}$

$$\Rightarrow MD = (1+r)\left(-\frac{1}{P}\right)(1+r)^{-1} \sum_{i=1}^{n} \frac{(-i)C_i}{(1+r)^i} = \frac{1}{P} \sum_{i=1}^{n} \frac{iC_i}{(1+r)^i}$$

- For level-coupon bonds we find:

$$MD = \ldots = \frac{c(1+r)\left((1+r)^n - 1\right) + nr(r-c)}{cr\left((1+r)^n - 1\right) + r^2}$$