Stochastic Methods Lab, Jacobs University Bremen, Fall 2022 — Session 11, Oct. 13, 2022

Prof. Dr. Soeren Petrat

2.4 Binomial Tree and Calibration

We want to choose u, d, p_s in such a way that they match a given (observed) expectation value and variance, for large n (many steps).

We had $P(j,n) = \binom{n}{j} p_s^j (1-p_s)^{n-j}$, $S_T(j\,up) = S_0 u^j d^{n-j}$

Now let's put $S_T(j\,up) = S_0 e^{Y_j}$, $Y_j :=$ stock's rate of return

$$\Rightarrow \quad Y_j = \ln \frac{S_T(j\,up)}{S_0} = \ln u^j d^{n-j} = \ln \left(\frac{u}{d}\right)^j d^n \underset{\substack{\uparrow \\ \ln(ab) = \ln a + \ln b \\ \ln(a^x) = x \ln a}}{=} j \ln\left(\frac{u}{d}\right) + n \ln d$$

Next: compute expectation value and variance of Y_j (Y as a fct. of j).

<u>Def.:</u>

- Expectation value of x is $\mathbb{E}(x) = \sum_{j=0}^{n} x_j P(j,n)$
- Variance of x is $\mathrm{Var}(x) = \mathbb{E}\left(x - \mathbb{E}(x)\right)^2$

A few computational rules:

- $\mathbb{E}(x+y) = \mathbb{E}(x) + \mathbb{E}(y)$, $\mathbb{E}(\lambda x) = \lambda \mathbb{E}(x)$
- $\mathrm{Var}(x) = \mathbb{E}\left(x^2 - 2x\mathbb{E}(x) + \mathbb{E}(x)^2\right) = \mathbb{E}(x^2) - 2\mathbb{E}(x)\mathbb{E}(x) + \mathbb{E}(x)^2$

$$= \mathbb{E}(x^2) - \mathbb{E}(x)^2$$

- $\mathrm{Var}(\lambda x) = \lambda^2\, \mathrm{Var}(x)$

- $\mathrm{Var}(x+y) = \mathbb{E}((x+y)^2) - \left(\mathbb{E}(x+y)\right)^2$

$$= \underline{\mathbb{E}(x^2)} + 2\,\mathbb{E}(xy) + \underline{\mathbb{E}(y^2)} - \left(\underline{\mathbb{E}(x)^2} + 2\,\mathbb{E}(x)\mathbb{E}(y) + \underline{\mathbb{E}(y)^2}\right)$$

$$= \underline{\mathrm{Var}(x)} + \underline{\mathrm{Var}(y)} + 2\underbrace{\left(\mathbb{E}(xy) - \mathbb{E}(x)\mathbb{E}(y)\right)}$$

$= \mathrm{Cov}(x,y) =$ covariance; does not generally vanish, but is zero for independent x and y

Next: y_j is a linear fct. of j, so we need to compute $\mathbb{E}$ and Var of $x_j = j$ (the identity fct.), i.e., $\underbrace{\mathbb{E}(x) \equiv \mathbb{E}(x_j) \equiv \mathbb{E}(j) \equiv \mathbb{E}(\mathbb{1}) \equiv \mathbb{E}(id)}_{\text{different notations for same object}}$.

$$\mathbb{E}(x) = \sum_{j=0}^{n} j\, P(j,n) = \sum_{j=0}^{n} j \binom{n}{j} p_s^j (1-p_s)^{n-j}$$

$$= (1-p_s)^n \sum_{j=0}^{n} j \binom{n}{j} \left(\frac{p_s}{1-p_s}\right)^j$$

note: $\sum_{j=0}^{n} j \binom{n}{j} z^j$ can be computed by shifting summation indices or via derivatives:

$$z \frac{d}{dz} \underbrace{\sum_{j=0}^{n} \binom{n}{j} z^j}_{=(1+z)^n} = z \sum_{j=0}^{n} j \binom{n}{j} z^{j-1} = \sum_{j=0}^{n} j \binom{n}{j} z^j$$

$$\Rightarrow \sum_{j=0}^{n} j \binom{n}{j} z^j = z \frac{d}{dz}(1+z)^n = z n (1+z)^{n-1}$$

$$\Rightarrow \mathbb{E}(x) = (1-p_s)^n \left(\frac{p_s}{1-p_s}\right) n \underbrace{\left(1+\frac{p_s}{1-p_s}\right)^{n-1}}_{\left(\frac{1}{1-p_s}\right)^{n-1}} \qquad \left(z = \frac{p_s}{1-p_s}\right)$$

$$= \left(\frac{p_s}{1-p_s}\right) n \, (1-p_s)$$

$$= n p_s$$

by similar computation: $\mathbb{E}(x^2) = n p_s \left((n-1) p_s + 1 \right)$

$$\Rightarrow \mathrm{Var}(x) = \mathbb{E}(x^2) - \mathbb{E}(x)^2 = n p_s - n p_s^2 = n p_s (1-p_s)$$

Now we can compute $\mathbb{E}$ and Var of $Y_j = j \ln\left(\frac{u}{d}\right) + n \ln d$

We find:

- $\mathbb{E}(Y) = \mathbb{E}\left(j \ln\left(\frac{u}{d}\right) + n \ln d\right) = \ln\frac{u}{d}\, \mathbb{E}(j) + n \ln d = \left(\ln\frac{u}{d}\right) n p_s + n \ln d$

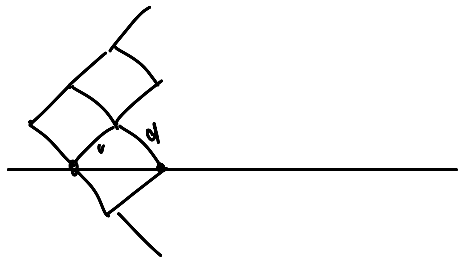
- $\mathrm{Var}(Y) = \mathrm{Var}\left(j \ln\frac{u}{d} + n \ln d\right) \underset{\substack{\uparrow \\ \mathrm{Cov}(\mathrm{const}, x) = 0 \\ \mathrm{Var}(\mathrm{const}) = 0}}{=} \left(\ln\frac{u}{d}\right)^2 \mathrm{Var}(j) = \left(\ln\frac{u}{d}\right)^2 n p_s (1-p_s)$

Next, we want to match $\mathbb{E}(Y)$ and $\mathrm{Var}(Y)$ to given values:

$$\mathbb{E}(Y_j) \xrightarrow{n\to\infty} \mu T \qquad , \mathrm{Var}(Y_j) \xrightarrow{n\to\infty} \sigma^2 T$$

μ is called mean-value $\qquad$ σ is called volatility

There is one more sensible condition: $ud = 1$



Same value for stock price if $ud=1$ on all horizontal lines

Then
- $\mathbb{E}(Y) = \left(\ln \frac{u}{d}\right) n p_s + n \ln d = 2(\ln u) n p_s - n \ln u = n(\ln u)(2p_s - 1)$
- $\operatorname{Var}(Y) = \left(\ln \frac{u}{d}\right)^2 n p_s (1-p_s) = 4(\ln u)^2 n p_s (1-p_s)$

Still there are several possible choices for u and p_s, a common one is

$$\boxed{p_s = \frac{1}{2} + \frac{1}{2}\frac{\mu}{\sigma}\sqrt{\frac{T}{n}} \quad , \quad u = e^{\sigma\sqrt{\frac{T}{n}}}} \qquad \left(d = e^{-\sigma\sqrt{\frac{T}{n}}}\right)$$

Check that they indeed give the right $\mathbb{E}$ and Var for large n:

- $\mathbb{E}(Y_j) = n\sigma\sqrt{\frac{T}{n}}\,\frac{\mu}{\sigma}\sqrt{\frac{T}{n}} = \mu T$
- $\operatorname{Var}(Y_j) = 4\left(\sigma\sqrt{\frac{T}{n}}\right)^2 n \left(\frac{1}{2} + \frac{1}{2}\frac{\mu}{\sigma}\sqrt{\frac{T}{n}}\right)\left(\frac{1}{2} - \frac{1}{2}\frac{\mu}{\sigma}\sqrt{\frac{T}{n}}\right)$

$$= \sigma^2 T \left(1 + \frac{\mu}{\sigma}\sqrt{\frac{T}{n}}\right)\left(1 - \frac{\mu}{\sigma}\sqrt{\frac{T}{n}}\right)$$

$$= \sigma^2 T - \underbrace{\sigma^2 T\left(\frac{\mu}{\sigma}\sqrt{\frac{T}{n}}\right)^2}_{\to 0 \text{ as } n\to\infty} \xrightarrow{n\to\infty} \sigma^2 T$$

Note again: μ and σ can be read off from real data (we will talk about this later),

therefore we want to choose $u, d, \underbrace{p_S}_{\text{not needed for option pricing}}$ depending on μ, σ.

Note: For the choice above, u does not depend on μ, only on σ

$\Rightarrow$ our option price is independent of μ!