Stochastic Methods Lab, Jacobs University Bremen, Fall 2022 — Session 15, Oct. 27, 2022

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Last time: We considered $X \sim \mathcal{N}(0,1)$, and $X = \sum_{i=1}^{n} X_i$ with X_i iid.

This implied $X_i \sim \mathcal{N}(0, \frac{1}{\sqrt{n}}) = \frac{1}{\sqrt{n}} \mathcal{N}(0,1)$

With $\frac{1}{n} = \Delta t$, we have $X_i \sim \sqrt{\Delta t}\, \mathcal{N}(0,1)$

This motivates the following definition:

<u>Def.:</u> A stochastic process $t \longmapsto W(t)$ for $t \geq 0$ is called $\boxed{\text{Brownian Motion (BM)}}$ or $\boxed{\text{Wiener process}}$ if:

$t \in \mathbb{R}$; for fixed t, $W(t)$ is a random variable

a) $W(0) = 0$ (convention)

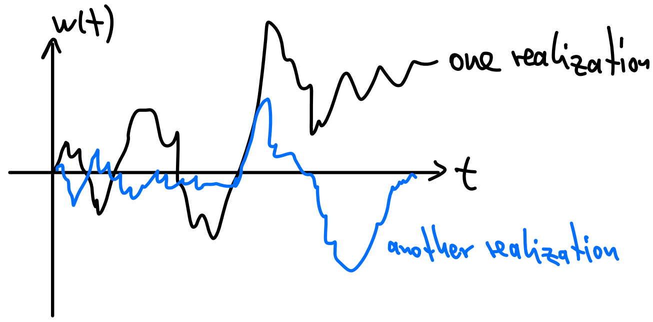
b) each realization is continuous,

c) for any $0 \leq s_1 < s_2 < t_1 < t_2$ the increments $W(t_2) - W(t_1)$ and $W(s_2) - W(s_1)$ are independent,

d) $W(t_2) - W(t_1)$ is distributed according to $\sqrt{t_2 - t_1}\, \mathcal{N}(0,1)$ for all $0 \leq t_1 < t_2$.

Note: one can indeed show that such a process exists and is unique

More pictures in python next time:



Is that a good model for stock prices?

No: • BM can become negative

(• parameters similar to μ and σ in calibrated binomial tree are missing)

• stocks seem to grow exponentially on average

Solution (better stock price model):

We use Geometric Brownian Motion (GBM): $\boxed{S(t) = S(0)\, e^{(\mu-\frac{\sigma^2}{2})t + \sigma W(t)}}$.

It turns out (see HW sheet 6) that the calibrated binomial tree model converges for $n \to \infty$ indeed to GBM!

A few notes on python implementation of BM:

• BM: $W_0 = 0$

$W_1 = \sqrt{\Delta t} \cdot$ sample from $\mathcal{N}(0,1)$

$W_2 = W_1 + \sqrt{\Delta t} \cdot$ sample from $\mathcal{N}(0,1)$

...

in python: $dW = \text{normal}(0, 1, \text{size}=n) \cdot \sqrt{\Delta t} \longrightarrow$ vector

$W = \text{cumsum}(dW)$ (= vector containing entries of cumulative sum)

$$= \begin{pmatrix} dW[0] \\ dW[0]+dW[1] \\ \vdots \\ dW[0]+\dots+dW[n-1] \end{pmatrix}$$

$W = \underbrace{r\_[0, W]}$ (add 0 at time 0)

r_[a,b] appends row vector b to vector a

$(a = (a_0, a_1, \dots, a_K), b = (b_0, b_1, \dots, b_\ell) \Rightarrow r\_[a,b] = (a_0, a_1, \dots, a_K, b_0, b_1, \dots, b_\ell))$

- ensemble of BMs:

# of time steps

$dW = \text{normal}(0, 1, \text{size}=(M, N)) \cdot \sqrt{\Delta t}$

↳ # of samples

$W = \text{cumsum}(dW, \text{axis}=1)$

↳ cumulative sum over row entries (axis=0 would be columns)

add zero vector

Note:
- similarly one can use, e.g., mean(W, axis=0)

  ↳ mean over samples

  transpose to plot rows

- W[:10:,:] selects 10 sample paths (plot(t, W[:10:,:].T))
- seed(k) (k some number) gives you the same samples (same random numbers)

3.2 Stochastic Integrals

Later: want PDE with randomness = stochastic partial differential equation = SPDE

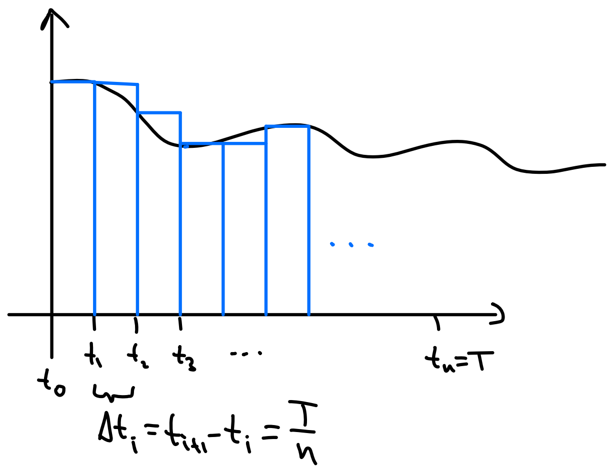
$\underbrace{\text{PDE}}_{\text{partial differential equation}}$

to model stock market: $dX = f\,dt + g\,dW$, W Brownian motion

In order to make sense of such an equation, we need to know how to define $\int g\,dW$ (bc. $\frac{dW}{dt}$ does not make sense: BM is not differentiable)

Recall Riemann sum for Riemann integral:

$$\int_0^T f(t)\,dt = \lim_{n\to\infty} \sum_{i=0}^{n-1} f(t_i)\,\Delta t_i$$



There are two kinds of stochastic integrals: Itô and Stratonovich

$\boxed{\text{Itô-integral:}}$

Defined analogously to Riemann sum:

$$\int_0^T f(t)\,dW(t) := \lim_{n\to\infty} \sum_{i=0}^{n-1} f(\underset{\substack{\uparrow \\ \text{evaluate } f \text{ at beginning} \\ \text{of interval}}}{t_i})\,\Delta W_i \quad \text{with } \overbrace{\Delta W_i}^{\text{random variable}} = W(t_{i+1}) - W(t_i) \overset{\substack{\text{distributed like} \\ \downarrow}}{\sim} \sqrt{t_{i+1} - t_i}\;\mathcal{N}(0,1)$$

$$= \sqrt{\Delta t}\;\mathcal{N}(0,1)$$

Stratonovich Integral:

evaluate f at middle of interval

$$\int_0^T f(t) \circ dW(t) = \lim_{n \to \infty} \sum_{i=0}^{n-1} f(t_i^*) \Delta W_i \qquad \text{with } t_i^* = \frac{t_{i+1} + t_i}{2}$$

notation to differentiate it from Itô integral