Stochastic Methods Lab, Jacobs University Bremen, Fall 2022 — Session 16, Nov. 2, 2022

Prof. Dr. Soeren Petrat

We continue our discussion of stochastic integrals.

Recall:

- Itô-integral:

$$\int_0^T f(t)\,dW(t) := \lim_{n\to\infty} \sum_{i=0}^{n-1} f(t_i)\,\Delta W_i \quad \text{with } \Delta W_i = W(t_{i+1}) - W(t_i) \sim \sqrt{\Delta t_i}\,\mathcal{N}(0,1)$$

$$(\Delta t_i = t_{i+1} - t_i)$$

- Stratonovich Integral:

$$\int_0^T f(t)\circ dW(t) = \lim_{n\to\infty} \sum_{i=0}^{n-1} f(t_i^*)\,\Delta W_i \quad \text{with } t_i^* = \frac{t_{i+1}+t_i}{2}$$

Note: One can show that this definition is equal to $\lim_{n\to\infty} \sum_{i=0}^{n-1} \left(\frac{f(t_{i+1})+f(t_i)}{2}\right)\Delta W_i$.

To illustrate their difference, consider the following example (see also HW 6 Problem 4):

Ex.: integrate Brownian motion against itself: $\int_0^T W(t)\,dW(t) = \int_0^T W\,dW$

Note: If $W(t)$ were differentiable, we could use the chain rule $dW = \frac{dW}{dt}\,dt$

$$\Rightarrow \int_0^T W(t)\,dW(t) = \int_0^T W(t)\,\frac{dW(t)}{dt}\,dt = \frac{1}{2}\int_0^T \frac{d}{dt}\left(W(t)^2\right)dt = \frac{1}{2}W(T)^2 - \frac{1}{2}\underbrace{W(0)^2}_{=0}$$

But $W(t)$ is not differentiable!

Let us compute "by hand":

$$\int_0^T W(t)\,dW(t) := \lim_{n\to\infty} \sum_{i=0}^{n-1} W(t_i)\,\Delta W_i = \lim_{n\to\infty} \sum_{i=0}^{n-1} W(t_i)\big(W(t_{i+1}) - W(t_i)\big)$$

$$= W(t_i)\,W(t_{i+1}) - W(t_i)^2$$

$$= \frac{1}{2}\Big[\, W(t_{i+1})^2 - W(t_i)^2 - \underbrace{\big(W(t_{i+1}) - W(t_i)\big)^2}_{= W(t_{i+1})^2 - 2W(t_{i+1})W(t_i) + W(t_i)^2} \Big]$$

$$\Rightarrow \int_0^T W(t)\,dW(t) = \frac{1}{2}\lim_{n\to\infty} \underbrace{\sum_{i=0}^{n-1} \big(W(t_{i+1})^2 - W(t_i)^2\big)}_{= W(T)^2 - W(0)^2 = W(T)^2 \quad \text{(telescope sum)}} - \frac{1}{2}\lim_{n\to\infty}\sum_{i=0}^{n-1} \Delta W_i^2$$

$$(W_1^2 - W_0^2) + (W_2^2 - W_1^2) + (W_3^2 - W_2^2) + \dots (W_n^2 - W_{n-1}^2) = W_n^2 - W_0^2$$

Question: How is $\sum_{i=0}^{n-1} \Delta W_i^2$ distributed?

We know that $\mathbb{E}(\Delta W_i^2) = \Delta t = \frac{T}{n}$ (recall that $\Delta W \sim \sqrt{\Delta t}\,\mathcal{N}(0,1)$)

$$\mathbb{E}(\Delta W_i^2) = \mathrm{Var}(\Delta W_i) = \mathbb{E}(\Delta W_i^2) - \underbrace{\mathbb{E}(\Delta W_i)^2}_{=0}$$

$$\Rightarrow \mathbb{E}\Big(\sum_{i=0}^{n-1} \Delta W_i^2\Big) = \sum_{i=0}^{n-1} \frac{T}{n} = T$$

What about variance?

$$\mathrm{Var}\Big(\sum_{i=0}^{n-1} \Delta W_i^2\Big) = \mathbb{E}\Big(\sum_{i,j=0}^{n-1} \Delta W_i^2 \Delta W_j^2\Big) - \underbrace{\mathbb{E}\Big(\sum_{i=0}^{n-1} \Delta W_i^2\Big)^2}_{T^2}$$

$$= \mathbb{E}\Big(\underbrace{\sum_{i=0}^{n-1} \Delta W_i^4}\Big) + \mathbb{E}\Big(\underbrace{\sum_{i\neq j} \Delta W_i^2 \Delta W_j^2}\Big) - T^2$$

ΔW_i and ΔW_j independent!

One can compute: $\approx \sum_{i=0}^{n-1} \frac{T^2}{n^2} = O\left(\frac{1}{n}\right)$ $\quad = \sum_{i\neq j} \mathbb{E}(\Delta W_i^2 \Delta W_j^2) = n(n-1)\frac{T^2}{n^2} = T^2 + O\left(\frac{1}{n}\right)$

$$= O\left(\frac{1}{n}\right)$$

$\Rightarrow$ Variance vanishes in the limit $n \to \infty$ $\Rightarrow \sum_{i=0}^{n-1} \Delta W_i^2 = T$, a constant

$\Rightarrow$ a deterministic process

$$\boxed{\text{Conclusion: } \int_0^T W(t)\,dW(t) = \frac{1}{2}W(T)^2 - \frac{1}{2}T}$$

(different from usual integral!)

Let's consider same example as before with the Stratonovich integral:

$$\int_0^T W(t) \circ dW(t) = \lim_{n\to\infty} \sum_{i=0}^{n-1} \underbrace{W(t_i^*)\big(W(t_{i+1}) - W(t_i)\big)}$$

$$\frac{1}{2}\Big[W(t_{i+1})^2 - W(t_i)^2 + \big(W(t_i^*) - W(t_i)\big)^2 - \big(W(t_{i+1}) - W(t_i^*)\big)^2 \Big]$$

Similar to before:

- $\mathbb{E}\left(\big(W(t_i^*) - W(t_i)\big)^2\right) = t_i^* - t_i = \frac{t_{i+1}+t_i}{2} - t_i = \frac{t_{i+1}-t_i}{2} = \frac{\Delta t}{2}$
- $\mathbb{E}\left(\big(W(t_{i+1}) - W(t_i^*)\big)^2\right) = t_{i+1} - \left(\frac{t_{i+1}+t_i}{2}\right) = \frac{\Delta t}{2}$

and variances vanishes

$$\Rightarrow \int_0^T W(t) \circ dW(t) = \frac{1}{2}W(T)^2 + \frac{T}{2} - \frac{T}{2} = \frac{1}{2}W(T)^2$$

(as we would expect from regular calculus)

In comparison:

- Stratonovich: • much "nicer", properties more similar to usual integration
  - but in each step function is evaluated in between t_{i+1} and t_i
  - ↳ undesirable for some SPDEs

- Itô: • technically a bit "harder" to handle, results different from regular calculus
  - but, at each t_i, f is evaluated and an increment is added
  - ↳ $\boxed{\text{this is what we want for stock market SPDE later}}$

Python hints for HW 6:

- GBM: $S(t) = S(0)\exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W(t)\right)$

$$= S(0)\exp\left(\left(\mu - \frac{\sigma^2}{2}\right)\sum_{i=1}^{N} \Delta t + \sigma \sum_{i=1}^{N} \Delta W_i\right)$$

$$= S(0) \prod_{i=1}^{N} \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)\Delta t + \sigma \Delta W_i\right)$$

$$= \underbrace{S(0)}_{=1 \text{ for Problem 2}} \text{cumprod}(\ldots)$$

- paths from binomial tree: hints: random_sample((M, N)) gives random sample from [0,1] with uniform probability

  (or look at "choice" fct.)

− Problem 3: only need $S(T)$ (need not generate full paths)

$$\left(S(T) = S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma W(T) \right) \right)$$

random variable $\sim \sqrt{T}\,\mathcal{N}(0,1)$

$(W(T) = W(T) - W(0))$

− Problem 4:

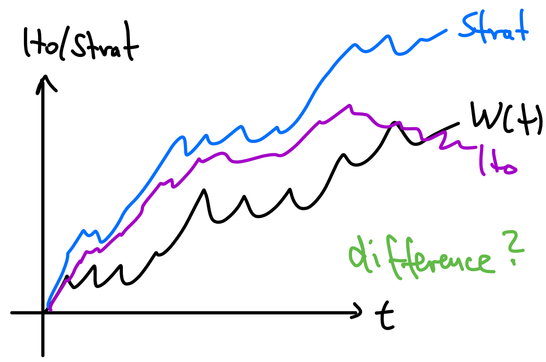
$$W = dW_0 + dW_1 + dW_2 + \dots$$
$$= (W_1 - W_0) + (W_2 - W_1) + (W_3 - W_2)$$

$$\Rightarrow \text{Ito} = W_0 \underbrace{(W_2 - W_0)}_{dW_0 + dW_1} + W_2 \underbrace{(W_4 - W_2)}_{dW_2 + dW_3} + \dots$$

$$\Rightarrow \text{Strat} = W_1 (W_2 - W_0) + W_3 (W_4 - W_2) + \dots$$

(recall W[a:b:inc] notation)

For exercise, just look at one realization:



Use cumsum to be able to plot $\int_0^t W(s)\,ds$ against t.