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For the more general version of the Itô formula, we consider the class of stochastic processes that are solutions to

$$\boxed{dX(t) = f(X(t),t)\,dt + g(X(t),t)\,dW(t)}$$

Solutions $X(t)$ are called $\boxed{\text{Itô-process.}}$

For example: $dS = \mu S\,dt + \sigma S\,dW$ (GBM) is an Itô process

So now, we want to look at $F(X(t),t)$ and find an expression for dF.

Similar to before:

$$dF(X,t) = \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial X}\underbrace{dX} + \frac{1}{2}\frac{\partial^2 F}{\partial X^2}\underbrace{(dX)^2} + \underbrace{\frac{1}{2}\frac{\partial^2 F}{\partial t^2}(dt)^2 + \frac{\partial^2 F}{\partial X \partial t}dX\,dt}_{\to 0 \text{ if integrated}}$$

$dX = f\,dt + g\,dW$

$$(dX)^2 = (f\,dt + g\,dW)^2 = \underbrace{f^2(dt)^2 + 2fg\,dt\,dW}_{\to 0 \text{ if integrated}} + g^2\underbrace{(dW)^2}_{\to dt}$$

$\left(\sum_{i=1}^{n}\frac{1}{n^2} \sim \frac{1}{n} \to 0\right)$ $\quad\left(\sum_{i=1}^{n}\frac{1}{n}\frac{1}{\sqrt{n}} \sim \frac{1}{\sqrt{n}} \to 0\right)$

$$\Rightarrow \boxed{dF = \left[\frac{\partial F}{\partial t} + \frac{\partial F}{\partial x} f + \frac{1}{2}\frac{\partial^2 F}{\partial x^2} g^2\right] dt + g\frac{\partial F}{\partial x} dW}$$

This is called $\boxed{\text{Itô lemma.}}$

Remark: For $f=0$, $g=1$, we get $X=W$, and in this case

$$dF = \left[\frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}\right] dt + \frac{\partial F}{\partial x} dW \quad \text{(just as before)}$$

Let us now discuss the application of Itô's lemma to GBM:

We defined GBM as $S(W(t),t) = e^{(\mu-\frac{\sigma^2}{2})t + \sigma W(t)}$ $\quad (S_0 = 1)$

$\left(S(x,t) = e^{(\mu-\frac{\sigma^2}{2})t+\sigma x}\right)$

With Itô, it satisfies the SDE $dS = \left[\frac{\partial S}{\partial t} + \frac{1}{2}\frac{\partial^2 S}{\partial x^2}\right]dt + \frac{\partial S}{\partial x} dW$

$$= \left[(\mu-\tfrac{\sigma^2}{2})S + \tfrac{1}{2}\sigma^2 S\right]dt + \sigma S\, dW$$

$$\Rightarrow \boxed{dS = \underbrace{\mu S}_{f}\, dt + \underbrace{\sigma S}_{g}\, dW}$$

With that, we can, e.g., compute $\mathbb{E}(S(t)^n)$.

Take $F(S(t),t) = S(t)^n$ and apply Itô lemma:

$$\Rightarrow dS^n = \left[\frac{\partial F}{\partial t} + \frac{\partial F}{\partial x} f + \frac{1}{2} \frac{\partial^2 F}{\partial x^2} g^2 \right] dt + g \frac{\partial F}{\partial x} dW$$

$$= \left[0 + n S^{n-1} \mu S + \frac{1}{2} n(n-1) S^{n-2} (\sigma S)^2 \right] dt + \sigma S n S^{n-1} dW$$

$$= \underbrace{\left[n\mu + \frac{1}{2} \sigma^2 n(n-1) \right] S^n}_{f_n(S^n)} dt + \underbrace{\sigma n S^n}_{g_n(S^n)} dW$$

$\Rightarrow S^n$ is again a GBM with different parameters!

$$\Rightarrow \mathbb{E}(S(T)^n) \underbrace{- \mathbb{E}(S(0)^n)}_{=1} = (n\mu + \tfrac{1}{2}\sigma^2 n(n-1)) \int_0^T \mathbb{E}(S(t)^n)\, dt + \sigma n \int_0^T \underbrace{\mathbb{E}(S(t)^n dW(t))}_{=0}$$

$$\left(\mathbb{E}(S(t)^n) = k(t) \Rightarrow k(T) - k(0) = (n\mu + \tfrac{1}{2}\sigma^2 n(n-1)) \int_0^T k(t)\, dt, \text{ or } \frac{dk}{dt} = (n\mu + \tfrac{1}{2}\sigma^2 n(n-1))\, k \right)$$

$$\Rightarrow \mathbb{E}(S(t)^n) = e^{[n\mu + \frac{1}{2}\sigma^2 n(n-1)]t}$$

In particular:

- $\mathbb{E}(S(t)) = e^{\mu t}$
- $\mathrm{Var}(S(t)) = \mathbb{E}(S(t)^2) - \mathbb{E}(S(t))^2$

$$= e^{(2\mu + \sigma^2)t} - e^{2\mu t}$$

$$= e^{2\mu t}(e^{\sigma^2 t} - 1) \quad (\approx \sigma^2 t \text{ for small } t)$$