Jacobs University
Spring 2022

March 1, 2022

Advanced Calculus and Methods of Mathematical Physics

Homework 4

Due on March 8, 2022

Problem 1 [**4 points**]

Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^2} & \text{when } (x,y) \neq (0,0)\,, \\ 0 & \text{when } (x,y) = (0,0)\,. \end{cases}$$

(a) Compute the directional derivative $D_{\boldsymbol{u}} f|_{(0,0)}$ for every $\boldsymbol{u} = (u_1, u_2) \in \mathbb{R}^2$ with $\|\boldsymbol{u}\| = 1$. Is $\boldsymbol{u} \mapsto D_{\boldsymbol{u}} f|_{(0,0)}$ linear?

(b) Show that f is continuous, but not differentiable at the origin.

Problem 2 [**4 points**]

Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \dfrac{x^3\,y}{x^6+y^2} & \text{when } (x,y) \neq (0,0)\,, \\ 0 & \text{when } (x,y) = (0,0)\,. \end{cases}$$

(a) Compute the directional derivative $D_{\boldsymbol{u}} f|_{(0,0)}$ for every $\boldsymbol{u} = (u_1, u_2) \in \mathbb{R}^2$ with $\|\boldsymbol{u}\| = 1$. Is $\boldsymbol{u} \mapsto D_{\boldsymbol{u}} f|_{(0,0)}$ linear?

(b) Show that f is not continuous at the origin.

Problem 3 [**4 points**]

Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \dfrac{x^2\,y}{x^4+y^2}\,\sqrt{x^2+y^2} & \text{when } (x,y) \neq (0,0)\,, \\ 0 & \text{when } (x,y) = (0,0)\,. \end{cases}$$

(a) Compute the directional derivative $D_{\boldsymbol{u}} f|_{(0,0)}$ for every $\boldsymbol{u} = (u_1, u_2) \in \mathbb{R}^2$ with $\|\boldsymbol{u}\| = 1$. Is $\boldsymbol{u} \mapsto D_{\boldsymbol{u}} f|_{(0,0)}$ linear?

(b) Show that f is continuous, but not differentiable at the origin.

Problem 4 [4 points]
Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be differentiable on $\mathbb{R}^2 \setminus \{0\}$. Let

$$h(r,\theta) = (r\,\cos\theta, r\,\sin\theta)$$

denote the change from polar to Cartesian coordinates and set $g = f \circ h$. Prove that, for $r > 0$,

$$\|(\nabla f) \circ h\|^2 = \left(\frac{\partial g}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial g}{\partial \theta}\right)^2 .$$

Problem 5 [4 points]
Let $\mathrm{Mat}(n \times n)$ denote the set of real $n \times n$ matrices. We consider the squaring map

$$S : \mathrm{Mat}(n \times n) \to \mathrm{Mat}(n \times n),\ S(A) \mapsto A^2.$$

In analogy to what we have defined in class, the map S is differentiable at A if there exists a linear map $DS|_A$ such that

$$S(A+H) = S(A) + DS|_A H + r_A(H), \quad \text{with } \lim_{H\to 0} \frac{\|r_A(H)\|}{\|H\|} = 0.$$

Here, $\|H\|$ denotes the operator norm of H. Show that A is differentiable everywhere and compute its derivative $DS|_A$.

Bonus Problem [4 points]: Banach fixed-point theorem
In any metric space (X,d), a map $f : X \to X$ is called a *contraction* if there is an $0 \le r < 1$ such that $d(f(x), f(y)) \le rd(x,y)$ for all $x, y \in X$. Prove the following theorem called *Banach fixed-point theorem* or *contraction mapping principle*: If (X,d) is a complete metric space, then any contraction $f : X \to X$ has a unique fixed point (i.e., a unique $x^* \in X$ with $f(x^*) = x^*$).

Hint: Uniqueness is easy. The fixed point can be constructed by defining a sequence $x_{n+1} = f(x_n)$. *Is this a Cauchy sequence?*