Jacobs University
Spring 2022

March 15, 2022

Advanced Calculus and Methods of Mathematical Physics

Homework 6

Due on March 22, 2022

Problem 1 [3 points]
(Kantorovitz, p. 78, Exercise 6. Warm-up.) Let $f \colon \mathbb{R}^k \to \mathbb{R}^m$ be defined by

$$f(x) = \sum_{i=1}^{k} (x_i, x_i^2, \dots, x_i^m) \,.$$

Compute the derivative $Df|_x$.

Problem 2 [5 points]
(From Rudin, Exercise 9.17.) Let $f \colon \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$f(x) = \begin{pmatrix} e^{x_1} \, \cos x_2 \\ e^{x_1} \, \sin x_2 \end{pmatrix} \,.$$

(a) What is the range of f?

(b) Show that the *Jacobian determinant*, $\det Df|_x$, is non-zero for every $x \in \mathbb{R}^2$. Thus every point in $\mathbb{R}^2$ has a neighborhood in which f is one-to-one. Nevertheless, f is not one-to-one on $\mathbb{R}^2$.

(c) Put $a = (0, \pi/3)$ and $b = f(a)$. Find an explicit formula for f^{-1}, compute $Df|_a$ and $Df^{-1}|_b$, and verify the formula for the derivative of the inverse from the statement of the inverse function theorem (i.e., $Df^{-1}|_b = (Df|_a)^{-1}$).

(d) What are the images under f of lines parallel to the coordinate axes?

Problem 3 [4 points]
(Kantorovitz, p. 106, Exercise 1.) Show that the equation

$$x^5 + y^5 + z^5 = 2 + xyz$$

determines in a neighborhood of the point $(1, 1, 1)$ a unique function $z = z(x, y)$ of class C^1, and calculate its partial derivatives with respect to x and y at the point $(1, 1)$.

Problem 4 [4 points]
Consider the equation

$$\sqrt{x^2 + y^2 + 2\,z^2} = \cos z$$

near $(0, 1, 0)$. Can you solve for x in terms of y and z? For z in terms of x and y?

Problem 5 [4 points]
Show that if r is a simple root of the polynomial

$$p(x) = a_0 + a_1\,x + \cdots + a_n\,x^n\,,$$

then r is a C^1 function of the coefficients $a_0, \ldots, a_n$.