Advanced Calculus and Methods of Mathematical Physics Session 3

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Last time: For $f \in C^{(n)}$, we write $f(x) = \underbrace{\sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k}_{=: P_{n,c}(x)} + \underbrace{R_{n,c}(x)}_{o(|x-c|^k)}$.

Note: Taylor polynomials are unique: Let f be n times differentiable, and $c \in I$. If Q is a polynomial of degree $\leq n$ with $f(x) - Q(x) = o((x-c)^n)$ as $x \to c$, then $P_{n,c}(x) = Q(x)$.

$\Rightarrow$ We can use any method to find a polynomial Q of degree $\leq n$ with $f(x) = Q(x) + o((x-c)^n)$; then $Q(x)$ is always the Taylor polynomial.

(Useful, e.g., for products or compositions of fct.s) ($\rightarrow$ see HW 1, Problem 3)

So far we have not made the remainder $R_{n,c}(x)$ very explicit. But using the fundamental thm. of calculus we can write:

$$f(x) - f(c) = \underbrace{\int_c^x f'(t)\,dt}$$

int. by parts with $(t-x)$ as antiderivative of 1 (w.r.t. t)

$$= \int_c^x 1 \cdot f'(t)\,dt = \underbrace{(t-x)f'(t)\Big|_c^x}_{=(x-c)f'(c)} - \int_c^x (t-x) f''(t)\,dt$$

$$\Rightarrow f(x) = f(c) + f'(c)(x-c) + \int_c^x (x-t) f''(t)\,dt$$

[check the next order or do a proof by induction]

Repeating this yields **Taylor's thm.**:

Let $f \in C^{(n+1)}(I)$, $I = (a,b)$, $c \in I$. If $x \in I$, then

$$R_{n,c}(x) = \int_c^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)\,dt \qquad \text{(integral form of remainder).}$$

This can be used for rigorous remainder estimates, e.g.:

- If additionally there is a $C>0$ s.t. $|f^{(n+1)}(x)| \le C \;\; \forall x \in I$, then
$$|R_{n,c}(x)| \le C \, \frac{|x-c|^{n+1}}{(n+1)!}$$

Note: There are also the Cauchy and Lagrange forms of the remainder.

(recall them, or see HW1 Problem 2 for the Lagrange form)

Next natural question:

Does $P_{n,c}(x) \longrightarrow f(x)$ as $n \to \infty$ around $x=c$? And in which sense?

$\Rightarrow$ Need to study sequences/series of functions and uniform convergence.

1.2 Sequences of Functions

Let $(f_n(x))_{n\in\mathbb{N}}$ be a sequence of functions $f_n : D \to \mathbb{R}$, $D \subset \mathbb{R}$.

An obvious notion of convergence is:

$\boxed{(f_n) \text{ converges to } f: D\to\mathbb{R} \text{ pointwise}} :\Longleftrightarrow f_n(x) \xrightarrow{n\to\infty} f(x)$ for all $x\in D$.

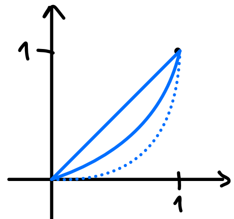
Things we want to know:

- If all f_n are continuous, is f also continuous?
- Is $\lim\limits_{n\to\infty} \int f_n(x)\,dx = \int \lim\limits_{n\to\infty} f_n(x)\,dx =: \int f(x)\,dx$?
- Is $\lim\limits_{n\to\infty} f_n'(x) = f'(x)$?

For affirmative answers, the notion of "pointwise convergence" is too weak.

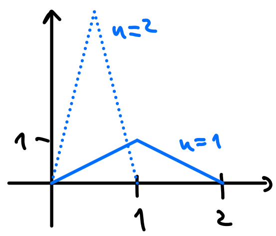
Examples:

- $f_n(x) = x^n$, $x \in [0,1]$



Each f_n is continuous, but $\lim_{n\to\infty} f_n(x) = \begin{cases} 0 & \text{for } x \in [0,1) \\ 1 & \text{for } x = 1 \end{cases}$ is not continuous.

- $f_n(x) = \begin{cases} n^2 x, & 0 \le x \le \frac{1}{n} \\ 2n - n^2 x, & \frac{1}{n} \le x \le \frac{2}{n} \\ 0, & \frac{2}{n} \le x \le 1 \end{cases}$



[check the computation]

Here, $\int_0^2 f_n(x)\,dx = \ldots = 1$ for all $n \in \mathbb{N}$, but $\lim_{n\to\infty} f_n(x) = 0$ for all $x \in [0,2]$.

So $\lim_{n\to\infty} \int_0^2 f_n(x)\,dx \neq \int_0^2 f(x)\,dx$.

- $f_n(x) = \frac{1}{n} \sin(nx)$.

Here, $f_n(x) \xrightarrow{n\to\infty} 0 = f(x)$, but $f_n'(x) = \cos(nx)$ does not have a limit $\forall x$ as $n \to \infty$.

Key concept to achieve nicer convergence results: uniform convergence

Recall: f_n conv. pointwise to f means:

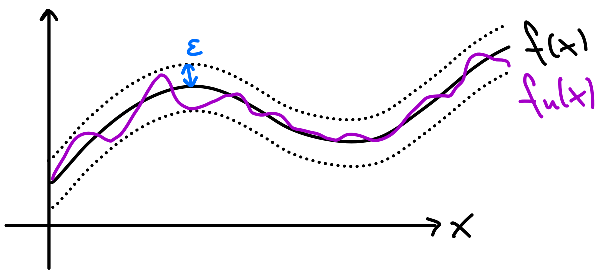
For all x we have: $\forall \varepsilon > 0 \; \exists N \in \mathbb{N}$ s.t. $\forall n \ge N$: $|f_n(x) - f(x)| < \varepsilon$

Definition:

$(f_n)_n$ with $f_n : D \to \mathbb{R}$, $D \subset \mathbb{R}$ $\boxed{\text{converges uniformly}}$ to $f : D \to \mathbb{R}$ if

$$\forall \varepsilon > 0 \ \underbrace{\exists N \in \mathbb{N}}_{\text{here, the } N \text{ large enough is independent of } x} \text{ s.t. } \forall n \geq N \text{ and } \forall x \in D : \ |f_n(x) - f(x)| < \varepsilon .$$

Graphically:



"all f_n's are in any ε-tube around f (for n large enough)"