Constructor University
Fall 2023

December 15, 2023

Foundations of Mathematical Physics

Final Exam

Instructions:

- Do all the work on this exam paper.
- Show your work, i.e., carefully write down the steps of your solution. You will receive points not just based on your final answer, but on the correct steps in your solution.
- No tools or other resources are allowed for this exam. In particular, no notes and no calculators.
- You are free to refer to any results proven in class or the homework sheets unless stated otherwise (and unless the problem is to reproduce a result from class or the homework sheets).

Name: ______________________________

Matric. No.: ______________________________

Problem 1: The Fourier Transform on Schwartz Space [25 points]

Let $\mathcal{S}(\mathbb{R}^d)$ be the Schwartz space as defined in class, and let $\mathcal{F}(f)(k) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x) e^{-ikx}\, dx$ denote the Fourier transform of a function $f \in \mathcal{S}(\mathbb{R}^d)$.

(a) In class, we defined the metric

$$d_{\mathcal{S}}(f, g) := \sum_{n=0}^{\infty} 2^{-n} \sup_{|\alpha|+|\beta|=n} \frac{\|f-g\|_{\alpha,\beta}}{1+\|f-g\|_{\alpha,\beta}}$$

for all $f, g \in \mathcal{S}(\mathbb{R}^d)$, where α and β are multi-indices.

(i) Define the semi-norms $\|\cdot\|_{\alpha,\beta}$ that are used in the definition of Schwartz space and its metric.

(ii) Prove that $d_{\mathcal{S}}(f, g) = 0$ is equivalent to $f = g$. (This is one of the properties of a metric.)

(b) What is the definition of completeness (of a metric space)? Is $(\mathcal{S}(\mathbb{R}^d), d_{\mathcal{S}})$ complete? (No proofs necessary here.)

(c) Prove that Fourier transform $\mathcal{F}$ is a continuous map from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}(\mathbb{R}^d)$.

(d) For polynomially bounded functions f we defined in class the pseudo-differential operator

$$f(-i\nabla) : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d), \psi(x) \mapsto \big(\mathcal{F}^{-1} M_f \mathcal{F} \psi\big)(x),$$

where M_f is the operator of multiplication with f. For $f(k) = e^{-iak}$ with $a \in \mathbb{R}^d$, compute $(f(-i\nabla)\psi)(x)$.

Problem 1: Extra Space

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Problem 2: Fourier Transform and Tempered Distributions [25 points]

Let $\mathcal{F}(f)(k) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x) e^{-ikx}\, dx$ denote the Fourier transform of a function f.

(a) Prove the Plancherel identity, i.e., that

$$\int_{\mathbb{R}^d} \left|(\mathcal{F}f)(k)\right|^2 \mathrm{d}k = \int_{\mathbb{R}^d} \left|f(x)\right|^2 \mathrm{d}x$$

for all $f \in \mathcal{S}(\mathbb{R}^d)$.

(b) Let $\psi_0 \in \mathcal{S}(\mathbb{R}^d)$ and consider the solution to the free Schrödinger equation

$$\psi : \mathbb{R}_t \to \mathcal{S}(\mathbb{R}^d), t \mapsto \psi(t) = \mathcal{F}^{-1} e^{-i\frac{k^2}{2}t} \mathcal{F}\psi_0.$$

Recall the formula

$$\mathcal{F}^{-1} e^{-i\frac{k^2}{2}t} \mathcal{F}\psi_0(x) = (2\pi i t)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{\frac{i(x-y)^2}{2t}} \psi_0(y)\, \mathrm{d}y.$$

(i) Prove that the L^2 norm of $\psi(t)$ is conserved, i.e., that

$$\|\psi(t)\|_{L^2(\mathbb{R}^d)} = \|\psi_0\|_{L^2(\mathbb{R}^d)}.$$

(ii) Prove that there is a constant $C > 0$ such that

$$t^{\frac{d}{2}} \|\psi(t)\|_{L^\infty(\mathbb{R}^d)} \leq C.$$

(c) Let $\mathcal{S}'(\mathbb{R}^d)$ be the space of tempered distributions, i.e., the dual space of $\mathcal{S}(\mathbb{R}^d)$. How is the Fourier transform of $T \in \mathcal{S}'(\mathbb{R}^d)$ defined?

(d) Let $f \in \mathcal{S}(\mathbb{R}^d)$ and define the tempered distribution $T_f \in \mathcal{S}'(\mathbb{R}^d)$ by $T_f(g) = \int_{\mathbb{R}^d} f(x) g(x)\, \mathrm{d}x$ for all $g \in \mathcal{S}(\mathbb{R}^d)$. Compute the Fourier transform of T_f.

(e) Let

$$f(x) := \begin{cases} x & , \text{for } x \geq 0 \\ 0 & , \text{for } x < 0, \end{cases}$$

and T_f the corresponding distribution (as defined in part (d)). Compute the distributional derivative of T_f.

Problem 2: Extra Space

Problem 2: Extra Space

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Problem 3: Bounded Operators [25 points]

(a) Let X and Y be normed spaces, and let $L : X \to Y$ be linear. Prove that L is continuous if and only if it is bounded.

(b) Let $(\varphi_n)_{n\in\mathbb{N}}$ be an orthonormal basis of a Hilbert space $\mathcal{H}$. We define a sequence $(A_n)_{n\in\mathbb{N}}$ of bounded linear operators in $\mathcal{H}$ by

$$A_n\psi := \sum_{k=1}^{\infty} \langle \psi, \varphi_k \rangle \, \varphi_{k+n}$$

for all $\psi \in \mathcal{H}$, where $\langle \cdot, \cdot \rangle$ is the scalar product on $\mathcal{H}$. Prove that $(A_n)_{n\in\mathbb{N}}$ converges weakly to zero, but not strongly.

Problem 3: Extra Space

Problem 3: Extra Space

Problem 3: Extra Space

Problem 4: Self-adjointness and Unitary Groups [25 points]

Let $\mathcal{L}(\mathcal{H})$ denote the space of bounded operators on a Hilbert space $\mathcal{H}$.

(a) Let $A \in \mathcal{L}(\mathcal{H})$ and let A^* be its Hilbert space adjoint. What is the definition of A self-adjoint? What is the definition of A symmetric? What is the relation between self-adjointness and symmetry for bounded operators?

(b) Define what a unitary operator is.

(c) Let H be a densly defined linear operator with domain $D(H) \subset \mathcal{H}$. Define what it means that H is the generator of a strongly continuous unitary one-parameter group $U(t)$.

(d) Let H with domain $D(H)$ be the generator of $U(t)$. Prove that

  (i) $U(t)D(H) = D(H)$ for all $t \in \mathbb{R}$,

  (ii) $\big[H, U(t)\big]\psi := HU(t)\psi - U(t)H\psi = 0$ for all $\psi \in D(H)$,

  (iii) H is symmetric, i.e., $\langle H\psi, \varphi\rangle = \langle \psi, H\varphi\rangle$ for all $\psi, \varphi \in D(H)$,

  (iv) U is uniquely determined by H and H is uniquely determined by U.

(e) Does every bounded linear operator H generate a unitary group? If no, give a counterexample; if yes, define the unitary group which is generated. (But no proofs are necessary here.)

Problem 4: Extra Space

Problem 4: Extra Space

Problem 4: Extra Space

Bonus Problem: Density Matrices and Mean-field Dynamics [25 points]

In class we defined for symmetric or antisymmetric $\Psi_N \in L^2(\mathbb{R}^{3N})$ the reduced one-particle density matrix γ_{Ψ_N} as the integral operator with integral kernel

$$\gamma_{\Psi_N}(x, y) = \int \mathrm{d}x_2 \dots \mathrm{d}x_N \, \overline{\Psi_N(y, x_2, \dots, x_N)} \Psi_N(x, x_2, \dots, x_N).$$

Furthermore, note that

$$\|A\|_{\mathcal{L}} = \sup_{\varphi \in L^2, \|\varphi\|=1} \langle \varphi, A\varphi \rangle$$

for all non-negative $A \in \mathcal{L}(L^2)$ (where A is called non-negative if $\langle \varphi, A\varphi \rangle \geq 0$ for all $\varphi \in L^2$).

(a) Let $\Psi_N \in L^2(\mathbb{R}^{3N})$ be symmetric in $x_1, \dots, x_N$ (bosonic wave function). Prove that the one-particle reduced density matrix γ_{Ψ_N} satisfies $\|\gamma_{\Psi_N}\|_{\mathcal{L}} \leq 1$. Give an example where $\|\gamma_{\Psi_N}\|_{\mathcal{L}} = 1$ and prove this property for your example.

(b) Let $\Psi_N \in L^2(\mathbb{R}^{3N})$ be antisymmetric in $x_1, \dots, x_N$ (fermionic wave function). Prove that the one-particle reduced density matrix γ_{Ψ_N} satisfies $\|\gamma_{\Psi_N}\|_{\mathcal{L}} \leq N^{-1}$. Give an example where $\|\gamma_{\Psi_N}\|_{\mathcal{L}} = N^{-1}$ and prove this property for your example.

(c) For $\varphi \in L^2(\mathbb{R}^3)$, let p^φ denote the projection onto the subspace spanned by φ, and let p_1^φ be this projection acting on the first variable only (as we defined in class). Prove that

$$\|\gamma_{\Psi_N} - p^\varphi\|_{\mathcal{L}} \leq 4\sqrt{1 - \langle \Psi_N, p_1^\varphi \Psi_N \rangle}.$$

Bonus Problem: Extra Space

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