

Foundations of Mathematical Physics

Homework 2

Due on Sep. 27, 2023, before the tutorial.

Problem 1 [5 points]: Integrals with Parameter

Prove the following lemma that was stated in class. Let $\Gamma \subset \mathbb{R}$ be an open interval and $f : \mathbb{R}^d \times \Gamma \rightarrow \mathbb{C}$ such that $f(x, \gamma) \in L^1(\mathbb{R}^d)$ for all fixed $\gamma \in \Gamma$. Let $I(\gamma) = \int_{\mathbb{R}^d} f(x, \gamma) dx$.

- (a) If the map $\gamma \mapsto f(x, \gamma)$ is continuous for almost all $x \in \mathbb{R}^d$ and if there is a $g \in L^1(\mathbb{R}^d)$ with $\sup_{\gamma \in \Gamma} |f(x, \gamma)| \leq g(x)$ for almost all $x \in \mathbb{R}^d$, then $I(\gamma)$ is continuous.
- (b) If the map $\gamma \mapsto f(x, \gamma)$ is continuously differentiable for all $x \in \mathbb{R}^d$ and if there is a $g \in L^1(\mathbb{R}^d)$ with $\sup_{\gamma \in \Gamma} |\partial_\gamma f(x, \gamma)| \leq g(x)$ for almost all $x \in \mathbb{R}^d$, then $I(\gamma)$ is continuously differentiable and

$$\frac{dI(\gamma)}{d\gamma} = \frac{d}{d\gamma} \int_{\mathbb{R}^d} f(x, \gamma) dx = \int_{\mathbb{R}^d} \frac{\partial}{\partial \gamma} f(x, \gamma) dx.$$

Problem 2 [5 points]: Integration by Parts in L^1

Let $f \in L^1(\mathbb{R}) \cap C^1(\mathbb{R})$ with $f' \in L^1(\mathbb{R})$.

- (a) Show that $f \in C_\infty(\mathbb{R})$.
- (b) Show integration by parts for $g \in L^\infty(\mathbb{R}) \cap C^1(\mathbb{R})$ with $g' \in L^\infty(\mathbb{R})$, i.e.,

$$\int_{\mathbb{R}} g(x) f'(x) dx = - \int_{\mathbb{R}} g'(x) f(x) dx.$$

- (c) Show that this implies that $\widehat{f}'(k) = ik\widehat{f}(k)$.

Problem 3 [5 points]: Smoothness and Decay of the Fourier Transform

Let $\ell \in \mathbb{N}$ and $f \in L^1(\mathbb{R})$. State (for each case separately) sufficient and preferably weak conditions for f which imply that

- (a) $\widehat{f} \in C^\ell(\mathbb{R})$,
- (b) $\sup_{k \in \mathbb{R}} \left| |k|^\ell \widehat{f}(k) \right| < \infty$,
- (c) $\widehat{f} \in L^1(\mathbb{R})$.

Problem 4 [5 points]: Fourier Transform of a Gaussian

Let $a \in \mathbb{C}$ with $\operatorname{Re} a > 0$ and $f : \mathbb{R} \rightarrow \mathbb{C}$ defined by $f(x) = e^{-ax^2/2}$. Show that $f \in \mathcal{S}(\mathbb{R})$ and

$$\widehat{f}(k) = a^{-1/2} e^{-k^2/(2a)}.$$