Constructor University October 4, 2023
Fall 2023

# Foundations of Mathematical Physics

## Homework 4

Due on Oct. 11, 2023, before the tutorial.

**Problem 1 [2 points]: Free Schrödinger Equation**
Finish the proof of Theorem 2.16 from class by showing that

$$\left(\mathcal{F}^{-1}e^{-i\frac{k^2}{2}t}\mathcal{F}\psi_0\right)(x) = (2\pi it)^{-\frac{d}{2}}\int e^{i\frac{(x-y)^2}{2t}}\psi_0(y)dy.$$

**Problem 2 [6 points]: Heat Equation**

(a) Let $\psi_0 \in \mathcal{S}(\mathbb{R}^d)$. Determine the solution to the heat equation

$$\begin{aligned} \partial_t\psi(t,x) &= \Delta_x\psi(t,x) \qquad \text{for all } (t,x) \in [0,\infty)\times\mathbb{R}^d, \\ \psi(0,x) &= \psi_0(x) \qquad \text{for all } x \in \mathbb{R}^d \end{aligned}$$

by using the Fourier transform. Write the solution as

$$\psi(t,x) = \int_{\mathbb{R}^d} K(t,x-y)\psi_0(y)dy, \tag{1}$$

and explicitly state what the function $K : (0,\infty)\times\mathbb{R}^d \to \mathbb{R}$ is.

(b) Let $\psi_0 \in C(\mathbb{R}^d)$ be bounded. Show that Equation (1) defines a bounded function $\psi \in C^\infty((0,\infty)\times\mathbb{R}^d)$ which solves the heat equation on $(0,\infty)\times\mathbb{R}^d$. Show also that $\psi$ can be continuously extended by $\psi_0$ at $t = 0$, i.e., show that $\lim_{t\to 0}\psi(t,x) = \psi_0(x)$ for all $x \in \mathbb{R}^d$. (*Hint: Use Problem 4 from Homework 2.*)

**Problem 3 [4 points]: Multiplication Operators on $L^p$**
Let $V : \mathbb{R}^d \to \mathbb{R}$ be measurable and $1 \le p \le \infty$. Show that $V$ defines a continuous multiplication operator

$$M_V : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d), \psi \mapsto V\psi$$

if and only if $V \in L^\infty(\mathbb{R}^d)$. Show that then

$$\|M_V\|_{\mathcal{L}(L^p)} := \sup_{\|f\|_{L^p}=1}\|M_V f\|_{L^p} = \|V\|_\infty.$$

**Problem 4 [8 points]: Convolution in $L^p$**
Let $1 \le p < \infty$ and $f \in L^p(\mathbb{R}^d)$.

(a) Using the Hölder inequality

$$\|fg\|_{L^1} \le \|f\|_{L^p}\|g\|_{L^q},$$

where $1/p + 1/q = 1$, show that for $g \in L^1(\mathbb{R}^d)$ we have

$$\|f * g\|_{L^p} \le \|f\|_{L^p}\|g\|_{L^1}.$$

*Hint: Show first that $f * g \in L^p(\mathbb{R}^d)$ by using $L^p(\mathbb{R}^d) = (L^q(\mathbb{R}^d))'$ (dual space of $L^q$, $1/p + 1/q = 1$). The inequality can then be shown to follow from this consequence of the Hahn-Banach theorem: For all $h \in L^p(\mathbb{R}^d)$ there is an $\tilde{h} \in L^q(\mathbb{R}^d)$ with $\|\tilde{h}\|_{L^q} = 1$ and*

$$\|h\|_{L^p} = \tilde{h}(h) := \int_{\mathbb{R}^d} \tilde{h}(x)h(x)dx.$$

(b) Let $\varphi \in C_c^\infty(\mathbb{R}^d)$ with $\varphi \ge 0$ and $\int_{\mathbb{R}^d} \varphi = 1$. Define $f_\sigma := f * (D_\sigma^1 \varphi)$ as in Problem 3 from Homework 2. Using (a), show that $f_\sigma$ converges to $f$ in $L^p(\mathbb{R}^d)$ as $\sigma \to 0$, i.e.,

$$\lim_{\sigma \to 0} \|f_\sigma - f\|_{L^p} = 0.$$

*Hint: Use that $C_c(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$ and Problem 4 from Homework 2.*