Constructor University — Fall 2023 — October 11, 2023

Foundations of Mathematical Physics

Homework 5

Due on Oct. 18, 2023, before the tutorial.

Problem 1 [4 points]: Distributional Derivatives
Let

$$g(x) := \begin{cases} x & ,\text{for } x \geq 0 \\ 0 & ,\text{for } x < 0, \end{cases}$$

and T_g the corresponding distribution. Compute all distributional derivatives of T_g (i.e., the derivative to arbitrary order).

Problem 2 [3 points]: Dilations ctd.
We continue Problem 3 from Homework 3. How does one have to define $\tilde{D}^p_\sigma : \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ in order to extend D^p_σ?

Problem 3 [3 points]: Fourier Transform
Compute the Fourier transform of the (first) distributional derivative of the delta distribution.

Problem 4 [4 points]: Cauchy Principal Part
The Cauchy principle part integral is defined as

$$\mathscr{P}\left(\frac{1}{x}\right) : f \mapsto \lim_{\varepsilon \downarrow 0} \int_{|x| \geq \varepsilon} \frac{1}{x} f(x)\,\mathrm{d}x$$

for any $f \in \mathcal{S}(\mathbb{R})$. Show that this is indeed a tempered distribution.

Problem 5 [3 points]: Uncertainty on $\mathcal{S}$
Prove the Heisenberg uncertainty principle. Let

$$(\delta x_j)^2 := \langle \psi, \big(x_j - \langle \psi, x_j \psi \rangle\big)^2 \psi \rangle, \qquad (\delta p_j)^2 := \langle \psi, \big(p_j - \langle \psi, p_j \psi \rangle\big)^2 \psi \rangle$$

be the variances in the position and the asymptotic momentum distributions, where $p_j = -i\partial_{x_j}$ and $\langle f, g \rangle = \int \overline{f} g$. (Let's just take $\psi \in \mathcal{S}$ here.) Prove that

$$\delta x_j \delta p_j \geq \frac{\|\psi\|^2}{2}.$$

Problem 6 [3 points]: Refined Uncertainty on $\mathcal{S}$
Prove the refined uncertainty principle (Hardy's inequality) on $\mathcal{S}(\mathbb{R}^3)$, i.e., that

$$\langle \psi, (-\Delta)\psi \rangle \geq \frac{1}{4} \langle \psi, |x|^{-2} \psi \rangle$$

for all $\psi \in \mathcal{S}(\mathbb{R}^3)$. *Hint: Look at the quantity* $\left[|x|^{-1} p_j |x|^{-1}, x_j \right]$, *where* $[A, B] := AB - BA$ *is the commutator. Note: This could be directly used to give a proof of the stability of hydrogen atoms.*