Constructor University
Fall 2023

October 18, 2023

Foundations of Mathematical Physics

Homework 6

Due on Oct. 25, 2023, before the tutorial.

Problem 1 [8 points]: Away from the support of $\widehat{\psi}$
Let $\psi_0 \in \mathcal{S}$ and let its Fourier transform have compact support, i.e., $K = \mathrm{supp}(\widehat{\psi})$ is compact. Let U be an open ε-neighborhood of K, i.e., the distance between the complement of U and K is ε, i.e., $\mathrm{dist}(U^c, K) = \varepsilon > 0$. Prove that then for any $m \in \mathbb{N}$ there is a constant $C_{m,\varepsilon}$ such that for any t, x with $x/t \notin U$ and $|t| \geq 1$,

$$\left| \left(e^{-it(-\Delta)/2} \psi_0 \right) (t, x) \right| \leq C_{m,\varepsilon} \left(1 + |t| \right)^{-m} .$$

Hint: One could write the phase factor as $e^{i\alpha S}$ with $S(k) = \frac{kx - k^2 t/2}{1+|t|}$ and some α. Prove that

$$e^{i\alpha S(k)} = \left[\frac{1}{i\alpha} |(\nabla S)(k)|^{-2} (\nabla S)(k) \nabla \right]^m e^{i\alpha S(k)},$$

and then integrate by parts.

Problem 2 [12 points]: Cauchy Principal Value continued
Recall from Homework Sheet 5 that the Cauchy principle part

$$\mathscr{P}\left(\frac{1}{x}\right) : \mathcal{S} \to \mathbb{C} : f \mapsto \lim_{\varepsilon \downarrow 0} \int_{|x| \geq \varepsilon} \frac{1}{x} f(x) \, \mathrm{d}x$$

is a tempered distribution.

(a) Prove that

$$\lim_{\varepsilon \downarrow 0} \frac{x - x_0}{(x - x_0)^2 + \varepsilon^2} = \mathscr{P}\left(\frac{1}{x - x_0}\right),$$

in the weak* sense.

(b) Let (φ_n) be a sequence of bounded functions on $\mathbb{R}$ so that $\int_{|x - x_0| \geq \varepsilon} \varphi_n(x) \, \mathrm{d}x \to 0$ as $n \to \infty$ for each $\varepsilon > 0$, $\varphi_n(x) \geq 0$, and $\int \varphi_n(x) \, \mathrm{d}x = c$ independent of n. Prove that $\varphi_n \to c\delta(x - x_0)$ in the weak* sense (i.e., φ_n is here regarded as a distribution).

(c) Prove that

$$\lim_{\varepsilon \to 0} \frac{\varepsilon}{(x - x_0)^2 + \varepsilon^2} = \pi \delta(x - x_0)$$

in the weak* sense.

(d) Prove the formula

$$\lim_{\varepsilon \downarrow 0} \frac{1}{x - x_0 + i\varepsilon} = \mathscr{P}\left(\frac{1}{x - x_0}\right) - i\pi\delta(x - x_0).$$

(e) Compute the Fourier transform of $\mathscr{P}\left(\frac{1}{x}\right)$. (*Hint: Use part (d).*)