Constructor University November 22, 2023
Fall 2023

Foundations of Mathematical Physics

Homework 11

Due on Nov. 30, 2023, before the tutorial.

Problem 1 [4 points]: Operator bounds

Let A, B be densely defined linear operators with $D(A) \subset D(B)$. Prove that the following two statements are equivalent:

(i) There are $a, b \geq 0$ such that

$$\|B\varphi\| \leq a\,\|A\varphi\| + b\,\|\varphi\|$$

for all $\varphi \in D(A)$.

(ii) There are $\tilde{a}, \tilde{b} \geq 0$ such that

$$\|B\varphi\|^2 \leq \tilde{a}^2\,\|A\varphi\|^2 + \tilde{b}\,\|\varphi\|^2$$

for all $\varphi \in D(A)$.

Prove also that the infimum over all permissible a in (i) coincides with the infimum over all permissible $\tilde{a}$ in (ii).

Problem 2 [8 points]: Self-adjointness

Let $V : \mathbb{R}^3 \to \mathbb{R}$ be such that $V \in L^2(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)$. Prove that then V is infinitesimally H_0-bounded, where $H_0 = -\frac{1}{2}\Delta$ with domain $D(H_0) = H^2(\mathbb{R}^3)$. With Kato-Rellich, this implies that $H = H_0 + V$ is self-adjoint on $D(H_0)$. Also use this result to prove that $H_0 + \frac{\lambda}{|x|}$ is self-adjoint on $H^2(\mathbb{R}^3)$ for all $\lambda \in \mathbb{R}$. *Hint: Prove that for all $a > 0$ there is a $b > 0$ such that for all $\varphi \in H^2(\mathbb{R}^3)$ we have $\|\varphi\|_{L^\infty} \leq a\|\Delta\varphi\|_{L^2} + b\|\varphi\|_{L^2}$.*

Problem 3 [8 points]: Projectors

For $\varphi \in L^2(\mathbb{R}^3)$ with $\|\varphi\| = 1$, we define for any $0 \leq k \leq N$ the bounded operator

$$P^{\varphi}_{N,k} := \sum_{\boldsymbol{a}\in\mathcal{A}_k} \prod_{j=1}^{N} (p^{\varphi}_j)^{1-a_j} (q^{\varphi}_j)^{a_j},$$

where

$$\mathcal{A}_k := \Big\{\boldsymbol{a} \in \{0,1\}^N : \sum_{j=1}^{N} a_j = k\Big\},$$

and p^{φ}_j, q^{φ}_j as defined in class. Prove the following properties:

(a) $P^{\varphi}_{N,k}$ is an orthogonal projector for all $0 \le k \le N$,

(b) $P^{\varphi}_{N,k} P^{\varphi}_{N,j} = 0$ for all $j \ne k$,

(c) $\sum_{k=0}^{N} P^{\varphi}_{N,k} = 1$,

(d) $\sum_{k=0}^{N} \frac{k}{N} P^{\varphi}_{N,k} = \frac{1}{N} \sum_{j=1}^{N} q^{\varphi}_{j}$.