2.3 Long-time Asymptotics and the Momentum Operator

Let us consider the solution $\psi(t) \in \mathcal{S}$ of the free SE, with $\psi(0) = \psi_0$.

Recall: probability that particle at time t is in $\Lambda \subset \mathbb{R}^d$ is $\mathbb{P}(X(t) \in \Lambda) = \int_\Lambda |\psi(t,x)|^2 dx$.

What about momentum (=velocity here, since mass $m=1$)? A-priori not defined in QM.

Let us consider the asymptotic velocity $= \frac{\text{distance}}{\text{time}}$ for large times t.

Probability that velocity is in $\Gamma \subset \mathbb{R}^d$ is $\mathbb{P}\left(\frac{X(t)}{t} \in \Gamma\right) = \mathbb{P}(X(t) \in t\Gamma) = \int_{t\Gamma} |\psi(t,x)|^2 dx$.

So next, we try to find $\lim_{t\to\infty} \mathbb{P}\left(\frac{X(t)}{t} \in \Gamma\right)$.

We compute:

$$\psi(t,x) := (2\pi i t)^{-\frac{d}{2}} \int e^{i\frac{(x-y)^2}{2t}} \psi_0(y)\, dy$$

$$= \frac{e^{i\frac{x^2}{2t}}}{(it)^{d/2}} (2\pi)^{-\frac{d}{2}} \int e^{-i\frac{x}{t}y} \left(e^{i\frac{y^2}{2t}} - 1 + 1\right) \psi_0(y)\, dy$$

$$= \frac{e^{i\frac{x^2}{2t}}}{(it)^{d/2}} \hat{\psi}_0\left(\frac{x}{t}\right) + \underbrace{\frac{e^{i\frac{x^2}{2t}}}{(it)^{d/2}} \hat{h}_t\left(\frac{x}{t}\right)}_{=: r(t,x)} \quad \text{, where } h_t(y) = \underbrace{\left(e^{i\frac{y^2}{2t}} - 1\right)}\psi_0(y).$$

$r(t,x)$ ← should be small for large t, since goes to 0 as $t\to\infty$

$|z+w|^2 = (z+w)(\bar z + \bar w) = |z|^2 + |w|^2 + \underbrace{\bar z w + z \bar w}_{=2\mathrm{Re}\,\bar z w}$

$$\Rightarrow \int_{t\Gamma} |\psi(t,x)|^2\,dx$$

$$= t^{-d}\int_{t\Gamma} \left|\hat\psi_0\left(\tfrac{x}{t}\right)\right|^2 dx + \overbrace{\int_{t\Gamma} |r(t,x)|^2\,dx}^{=t^{-d}\int_{t\Gamma}|\hat h_t(\frac{x}{t})|^2 dx} + 2\,\mathrm{Re}\; t^{-d}\int_{t\Gamma} \overline{\hat\psi_0\left(\tfrac{x}{t}\right)}\,\hat h_t\left(\tfrac{x}{t}\right) dx$$

change of variables $\frac{x}{t} = p$

$$= \int_\Gamma |\hat\psi_0(p)|^2\,dp + \underbrace{\int_\Gamma |\hat h_t(p)|^2\,dp} + 2\,\mathrm{Re}\,\underbrace{\int_\Gamma \overline{\hat\psi_0(p)}\,\hat h_t(p)\,dp}$$

$$\int_\Gamma |\hat h_t(p)|^2\,dp \le \int_{\mathbb{R}^d} |\hat h_t(y)|^2\,dy$$

Plancherel

$$= \int_{\mathbb{R}^d} |h_t(y)|^2\,dy$$

$$= \int \left|e^{i\frac{y^2}{2t}} - 1\right|^2 |\psi_0(y)|^2\,dy$$

$\xrightarrow{t\to\infty} 0$ by dominated convergence

(integrand $\to 0$ pointwise and bounded by $4|\psi_0| \in L^1$)

Cauchy-Schwarz

$$\int_\Gamma \overline{\hat\psi_0(p)}\,\hat h_t(p)\,dp \le 2\,\|\hat\psi_0(p)\|_{L^2}\,\underbrace{\|\hat h_t\|_{L^2}}_{\to 0 \text{ as } t\to\infty}$$

We thus have proven:

Theorem 2.42:

Let $\psi(t,x)$ be the solution to the free SE with initial condition $\psi_0 \in S$, let $\Gamma \subset \mathbb{R}^d$ be measurable. Then $\lim_{t\to\infty} \mathbb{P}\left(\frac{X(t)}{t} \in \Gamma\right) := \lim_{t\to\infty} \int_{t\Gamma} |\psi(t,x)|^2\,dx = \int_\Gamma |\hat\psi_0(p)|^2\,dp.$

Remarks:

- Recall $\psi(t_0,x) = \mathcal{F}^{-1} e^{-i\frac{k^2}{2}t_0} \mathcal{F}\psi_0(x)$, so $\hat\psi(t_0,k) = e^{-i\frac{k^2}{2}t_0}\hat\psi_0(k)$

  $\Rightarrow |\hat\psi(t_0,k)|^2 = |e^{-i\frac{k^2}{2}t_0}\hat\psi_0(k)| = |\hat\psi_0(k)|$, so Theorem 2.42 is independent of choice of initial time t_0.

  Also: For $\psi_{0,a}(x) := \psi_0(x-a) \overset{\text{see Session 10}}{=} (e^{-ia(-i\nabla)}\psi_0)(x)$, we find

  $|\widehat{\psi_{0,a}}(k)| = |\mathcal{F}\psi_{0,a}(k)| = |e^{-iak}\hat\psi_0(k)| = |\hat\psi_0(k)|$, so Theorem 2.42 is independent of translations.

- Expectation value of asymptotic momentum:

$$\mathbb{E} = \int p \, |\hat{\psi}_0(p)|^2 dp = \int \overline{\hat{\psi}_0(p)} \, p \, \hat{\psi}_0(p) dp = \int \overline{\psi(t,x)} \, (-i\nabla) \, \psi(t,x) dx$$

$$= \langle \psi_t, \hat{P} \psi_t \rangle$$ where $\hat{P} = -i\nabla_x$ is called "momentum operator".