Last time we proved the orthonormal decomposition:

$\psi = \psi_n + \psi_n^\perp$, with $\psi_n = \sum_{j=1}^n \langle \varphi_j, \psi \rangle \varphi_j$, $\{\varphi_j\}_{j=1,\dots,n}$ an ONS, where $\langle \psi_n, \psi_n^\perp \rangle = 0$, and thus $\langle \psi, \psi \rangle = \langle \psi_n + \psi_n^\perp, \psi_n + \psi_n^\perp \rangle = \langle \psi_n, \psi_n \rangle + \langle \psi_n^\perp, \psi_n^\perp \rangle$.

This implies the following inequalities:

- **Bessel inequality:** $\|\psi\|^2 \geq \sum_{j=1}^n |\langle \varphi_j, \psi \rangle|^2$ $\forall \psi \in \mathcal{H}, n \in \mathbb{N}$, follows directly from orthonormal decomposition ($\langle \psi, \psi \rangle \geq \langle \psi_n, \psi_n \rangle$).

- **Cauchy-Schwarz:** $|\langle \varphi, \psi \rangle| \leq \|\varphi\| \cdot \|\psi\|$ $\forall \varphi, \psi \in \mathcal{H}$, follows from Bessel for $n = 1$, $\varphi_1 = \frac{\varphi}{\|\varphi\|}$

- **Polarisation identity** for complex $\mathcal{H}$:

$$\boxed{\langle \varphi, \psi \rangle = \frac{1}{4}\left(\|\varphi+\psi\|^2 - \|\varphi-\psi\|^2 - i\|\varphi+i\psi\|^2 + i\|\varphi-i\psi\|^2 \right)} \quad \forall \varphi, \psi \in \mathcal{H}$$

(check by direct calculation) ↳ for real $\mathcal{H}$: $\langle \varphi, \psi \rangle = \frac{1}{4}\left(\|\varphi+\psi\|^2 - \|\varphi-\psi\|^2 \right)$

Next: some basic consequences of the concept of orthonormal basis.

Definition 3.7: A sequence $(\varphi_j)_j$ in $\mathcal{H}$ is called orthonormal basis (ONB) if $\psi = \sum_{j=1}^\infty \langle \varphi_j, \psi \rangle \varphi_j$ $\forall \psi \in \mathcal{H}$.

meaning $\left\| \psi - \sum_{j=1}^N \langle \varphi_j, \psi \rangle \varphi_j \right\|_{\mathcal{H}} \xrightarrow{N \to \infty} 0$

Note: With Zorn's lemma, every vector space has a basis (in the context of infinite dimensional $\mathbb{R}$ or $\mathbb{C}$ vector spaces called Hamel basis), meaning every vector can be written uniquely as a finite linear combination of basis vectors (recall linear Algebra). Thus, ONBs are a different notion (but obviously it makes a lot of sense to also call them basis).

Consequences (here, $\mathcal{H}$ is a Hilbert space and $(\varphi_j)_j$ an ONB):

The general definition of separability for topological spaces is the existence of a countable dense subset. By this definition, one can show that $\boxed{\text{a Hilbert space is separable iff it has an ONB}}$.

Proof: "$\Leftarrow$" $\left\{ \sum_{j=1}^{N} (a_j + i b_j)\varphi_j : N \in \mathbb{N}, a_j, b_j \in \mathbb{Q} \right\}$ is clearly a countable dense subset (of a complex Hilbert space)

"$\Rightarrow$" If $(\varphi_j)_j$ is any countable dense subset, we can – if necessary – just remove φ_i's such that the remaining $\{\varphi_j\}_{j\in J}$ are still $\underbrace{\text{linearly independent}}_{\text{recall that for } \infty\text{-dim. vector spaces, this means all finite linear combinations are linearly independent}}$, but still $\underbrace{\overline{\operatorname{span}\{\varphi_j\}_{j\in J}} = \mathcal{H}}_{\text{meaning } \psi = \sum_{j=1}^{\infty} a_j \varphi_j \ \forall \psi \in \mathcal{H}}$.

The remaining basis can be made orthonormal by $\underbrace{\text{Gram-Schmidt}}_{\text{recall this from linear Algebra}}$. $\square$

Notes:
- In this class we are only interested in separable Hilbert spaces
- Examples of non-separable spaces:

↳ more from physics: infinite spin chain: $\overbrace{\bigotimes_{k\in\mathbb{Z}}}^{\text{tensor product (rigorously introduced later)}} \mathbb{C}^2$ (reason: think of the two basis vectors in $\mathbb{C}^2$ as 0 and 1, then basis vectors in the infinite tensor product are all 0,1 sequences; but there are as many such sequences as real numbers)

↳ ℓ^∞ (bounded real sequences) is a non-separable Banach space

↳ more from math: space of almost periodic functions $H = \overline{X}$ (completion of X), where
$X = \{ f: \mathbb{R} \to \mathbb{C},\ f(t) = \sum_{j=1}^{n} c_j e^{is_j t},\ s_j \in \mathbb{R}, c_j \in \mathbb{C}, n \in \mathbb{N} \}$ with scalar product

$$\langle f, g \rangle = \lim_{N\to\infty} \frac{1}{2N} \int_{-N}^{N} \overline{f(t)}\, g(t)\, dt$$ ($\{e^{ist} : s \in \mathbb{R}\}$ is an uncountable orthonormal set)

A few consequences of ONBs:

- $\boxed{\text{\underline{Proposition 3.10:} An ONS } (\varphi_j)_j \text{ is an ONB iff: } \langle \varphi_j, \psi \rangle = 0\ \forall j \in \mathbb{N} \Rightarrow \psi = 0 \text{ (proof: HW)}}$

(This implies in particular that a reordering of an ONB is still an ONB.)

- $\boxed{\text{Parseval's identity: } \|\psi\|^2 = \sum_{j=1}^{\infty} |\langle \varphi_j, \psi \rangle|^2}$ for $(\varphi_j)_j$ an ONB

<u>Proof:</u> $\|\psi\|^2 = \langle \lim_{N\to\infty} \sum_{j=1}^{N} \langle \varphi_j, \psi \rangle \varphi_j , \lim_{M\to\infty} \sum_{i=1}^{M} \langle \varphi_i, \psi \rangle \varphi_i \rangle$

scalar product continuous

If $\lim_{n\to\infty} \|f_n - f\| = 0$, i.e., $f_n \to f$ in $\mathcal{H}$, then also $\lim_{n\to\infty} |\langle f_n, g\rangle - \langle f, g \rangle| \le \lim_{n\to\infty} \|f_n - f\| \|g\| = 0$, i.e., $\langle f_n, g \rangle \to \langle f, g \rangle$

$$= \lim_{N\to\infty} \lim_{M\to\infty} \underbrace{\langle \sum_{j=1}^{N} \langle \varphi_j, \psi \rangle \varphi_j , \sum_{i=1}^{M} \langle \varphi_i, \psi \rangle \varphi_i \rangle}$$

$\{\varphi_i\}$ ONB

$$= \sum_{j=1}^{N} \overline{\langle \varphi_j, \psi \rangle} \langle \varphi_j, \psi \rangle$$

$$= \lim_{N\to\infty} \sum_{j=1}^{N} |\langle \varphi_j, \psi \rangle|^2 \qquad \square$$

$\|\psi\|_{\mathcal{H}} = \|(\langle \varphi_j, \psi \rangle)_j\|_{\ell^2}$ — linear bijective map

- $\boxed{U: \mathcal{H} \to \ell^2,\ \psi \mapsto (\langle \varphi_j, \psi \rangle)_{j\in\mathbb{N}} \text{ is an isometric isomorphism}}$ (for separable $\mathcal{H}$) otherwise there would not be an ONB

<u>Proof:</u> Isometry due to Parseval $\|\psi\|_{\mathcal{H}}^2 = \sum_{j=1}^{\infty} |\langle \varphi_j, \psi \rangle|^2 = \|(\langle \varphi_j, \psi \rangle)_j\|_{\ell^2}^2 = \|U\psi\|_{\ell^2}^2$

injective clear bc. of isometry

$\underbrace{\text{Isomorphism since}}$ U surjective: for any $(c_j)_{j\in\mathbb{N}} \in \ell^2$, we choose $\psi = \sum_{j=1}^{\infty} c_j \varphi_j$.

Then $U\psi = \left(\langle \varphi_j, \sum_{k=1}^{\infty} c_k \varphi_k \rangle\right)_j = (c_j)_j$ ✓

(and $\psi \in \mathcal{H}$ since $\|\sum_{j=N}^{\infty} c_j \varphi_j\|_{\mathcal{H}}^2 = \langle \sum_{j=N}^{\infty} c_j \varphi_j, \sum_{i=N}^{\infty} c_i \varphi_i \rangle = \sum_{j=N}^{\infty} |c_j|^2 \xrightarrow{N\to\infty} 0$ $(c \in \ell^2)$). $\square$

$\Rightarrow$ All infinite dimensional separable Hilbert spaces are isometrically isomorphic to ℓ^2, and thus to each other. (Finite dimensional Hilbert spaces are isometrically isomorphic to $\mathbb{C}^n$.)

↳ ℓ^2 is the coordinate space for any separable Hilbert space (of infinite dimension) (any choice of ONB gives us an isometric isomorphism)

Example: $(\varphi_k)_{k\in\mathbb{Z}}$ with $\varphi_k = (2\pi)^{-\frac{1}{2}} e^{ikx}$ is an ONB for $L^2([0,2\pi])$;

$\psi = \sum_{k\in\mathbb{Z}} \langle \varphi_k, \psi \rangle \varphi_k$ is the Fourier series of ψ

Note: So why are we even interested in different Hilbert spaces? Because we are often interested in extra structure, e.g., operators on Hilbert spaces. Think of Fourier space, where differential operators become multiplication operators. (Or think of diagonalization in $\mathbb{C}^n$.)

Finally, we note that Hilbert spaces can be decomposed orthogonally.

Definition 3.14: For any $\overbrace{M \subset \mathcal{H}}^{\text{any subset}}$, we call $M^{\perp} := \{\psi \in \mathcal{H} : \langle \varphi, \psi \rangle = 0 \ \forall \varphi \in M\}$ the orthogonal complement of M.

Note: • $\mathcal{M} \cap \mathcal{M}^\perp = \begin{cases} \{0\} & \text{if } 0 \in \mathcal{M} \\ \emptyset & \text{if } 0 \notin \mathcal{M} \end{cases}$

• $\mathcal{M}^\perp$ is a $\underbrace{\text{closed}}_{\langle \varphi, \cdot \rangle \text{ continuous}}$ $\underbrace{\text{subspace}}_{\langle \varphi, \cdot \rangle \text{ linear}}$ of $\mathcal{H}$

Theorem 3.15: Let $\mathcal{M} \subset \mathcal{H}$ be a closed subspace. Then $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$, meaning $\forall \psi \in \mathcal{H}$ we have $\psi = \varphi + \varphi^\perp$ with unique $\varphi \in \mathcal{M}$, $\varphi^\perp \in \mathcal{M}^\perp$.