Operations Research, Constructor University Bremen, Fall 2023 — Session 4

Prof. Sören Petrat — Sep. 14, 2023

Last time: In standard form, the constraints can be written as $Ax = b$ (and $x \geq 0$).

Today: How do we find all solutions to $Ax = b$? (Afterwards: How do we select the optimal one?)

↳ Use Gaussian elimination to bring augmented matrix into reduced row echelon form:

Ex. of row echelon form: $\begin{pmatrix} 1 & 0 & 5 & 0 & 8 \\ 0 & 1 & 3 & 0 & 4 \\ 0 & 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

More exactly:

- All rows with just 0's are at the bottom. □
- Leading/first coefficient of a nonzero row is to the right of the leading coefficient from the row above. □
- All leading coefficients are 1 (they are called "pivots"). ○
- Each column with a leading 1 has 0's in all other entries. □

Ex. above: $\begin{pmatrix} 1 & 0 & 5 & 0 & 8 \\ 0 & 1 & 3 & 0 & 4 \\ 0 & 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

Let us do the explicit computations for an example: $A = \begin{pmatrix} 1 & 3 & 1 & 1 \\ 2 & 6 & 0 & -1 \end{pmatrix}$, $b = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\Rightarrow$ augmented matrix: $\left(\begin{array}{cccc|c} 1 & 3 & 1 & 1 & 2 \\ 2 & 6 & 0 & -1 & 1 \end{array}\right)$

Bring this into reduced row-echelon form using Gaussian elimination:

$-2R_1 + R_2 \to R_2$: $\left(\begin{array}{cccc|c} 1 & 3 & 1 & 1 & 2 \\ 0 & 0 & -2 & -3 & -3 \end{array}\right)$

$R_2/-2$: $\left(\begin{array}{cccc|c} 1 & 3 & 1 & 1 & 2 \\ 0 & 0 & 1 & \frac{3}{2} & \frac{3}{2} \end{array}\right)$ need this entry 0

$R_1 - R_2 \to R_1$: $\left(\begin{array}{cccc|c} 1 & 3 & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{3}{2} & \frac{3}{2} \end{array}\right)$

pivots

This corresponds to the two equations

$$x_1 + 3x_2 \quad - \frac{1}{2} x_4 = \frac{1}{2}$$
$$x_3 + \frac{3}{2} x_4 = \frac{3}{2}$$

$\Rightarrow$ We have two "free" variables, e.g., $x_4 = -\mu$, $x_2 = -\lambda$

$\hookrightarrow$ 2 (linearly independent) eq.s for 4 variables, i.e., $4-2=2$ variables can be chosen freely

$\Rightarrow$ $x_3 = \frac{3}{2} + \frac{3}{2}\mu$, $x_1 = \frac{1}{2} + 3\lambda - \frac{1}{2}\mu$

The solution to $Ax = b$ in this example is:

$$\Rightarrow \quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} + 3\lambda - \frac{1}{2}\mu \\ -\lambda \\ \frac{3}{2} + \frac{3}{2}\mu \\ -\mu \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{3}{2} \\ 0 \end{pmatrix}}_{\text{called } \boxed{\text{basic solution}}} + \underbrace{\lambda \begin{pmatrix} 3 \\ -1 \\ 0 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -\frac{1}{2} \\ 0 \\ \frac{3}{2} \\ -1 \end{pmatrix}}_{\text{the two vectors span the space of solutions to the homogeneous equation } Ax = 0}$$

$$\Rightarrow x = x^{basic} + x^{hom}, \text{ where } x^{hom} \text{ solves } Ax^{hom} = 0$$

We can use the following trick to immediately read off the solution:

$$\left(\begin{array}{cccc|c} 1 & 3 & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & \frac{3}{2} & \frac{3}{2} \\ 0 & 0 & 0 & -1 & 0 \end{array}\right) \begin{array}{l} \\ \leftarrow \text{add zero row} \\ \\ \leftarrow \text{add zero row} \end{array}$$

replace 0's on diagonal by -1

$$\Rightarrow \text{solution} = \left(\;\right) + \lambda \left(\;\right) + \mu \left(\;\right)$$

Recipe to get solutions directly from augmented matrix in row echelon form:

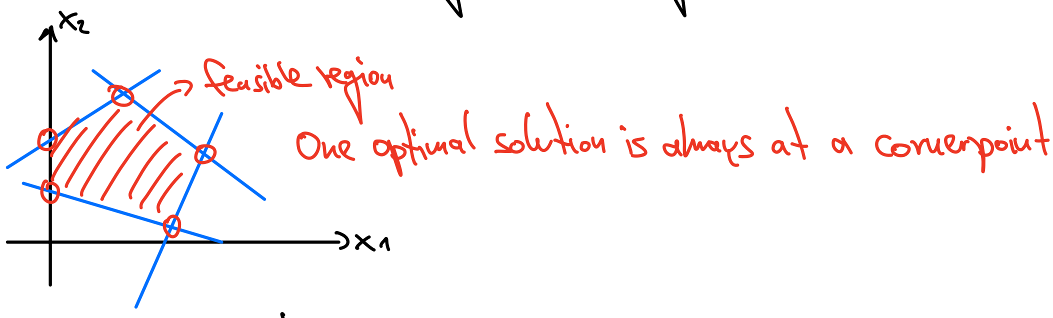
- add zero rows s.t. pivots, i.e., leading 1's, are on diagonal
- put -1 on diagonal in the zero rows
- read off solution as above

We call $\mathcal{B}$ the set of basic solution columns, i.e., the set containing the index of each column with a pivot. Here, $\mathcal{B} = \{1, 3\}$.
For a basic solution x^{basic} we have $x_j^{\text{basic}} = 0$ for $j \notin \mathcal{B}$.

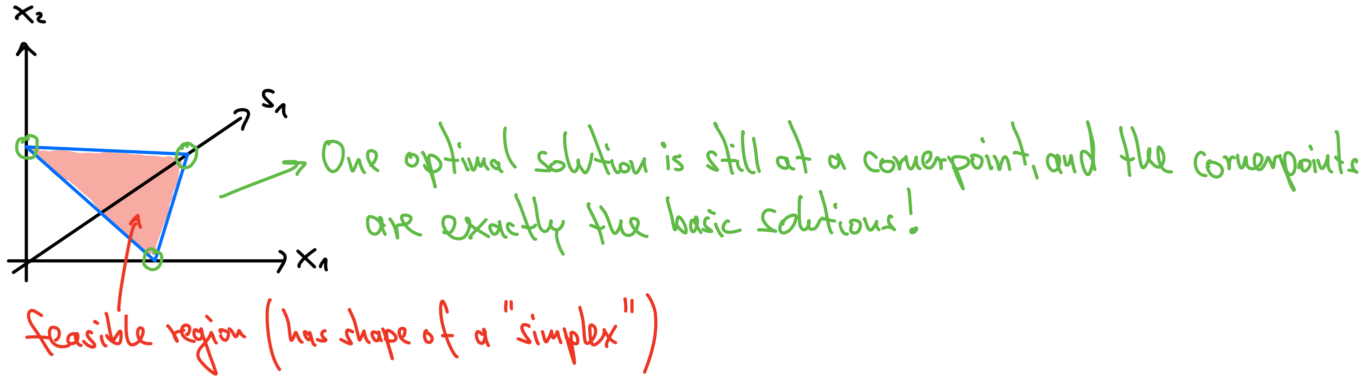
In fact, we are only interested into basic solutions here, because these are the cornerpoints of our feasible regions!

Why?

↳ For $Ax \leq b$ constraints, we got feasible regions such as this:



↳ For $Ax = b$ constraints:



Conclusion: Want to find all possible basic solutions.

In example above:

There are many ways to parametrize the solutions, e.g., also:

$$
\begin{array}{ccccc}
 & \text{pivot} & \text{columns} & & \\
 & \downarrow & \downarrow & & \\
\end{array}
$$

$$
\left(\begin{array}{cccc|c}
-1 & 0 & 0 & 0 & 0 \\
\frac{1}{3} & 1 & 0 & -\frac{1}{6} & \frac{1}{6} \\
0 & 0 & 1 & \frac{3}{2} & \frac{3}{2} \\
0 & 0 & 0 & -1 & 0
\end{array}\right)
$$

Note: Here we only care about pivot columns, i.e., it is ok if the leading coefficient is not 1.

(R_1 above $/3$)

i.e., $\mathcal{B} = \{2,3\}$

$$
\Rightarrow \quad x = \begin{pmatrix} 0 \\ \frac{1}{6} \\ \frac{3}{2} \\ 0 \end{pmatrix} + \tilde{\lambda} \begin{pmatrix} -1 \\ \frac{1}{3} \\ 0 \\ 0 \end{pmatrix} + \tilde{\mu} \begin{pmatrix} 0 \\ -\frac{1}{6} \\ \frac{3}{2} \\ -1 \end{pmatrix}
$$

Another possibility:

$$
\left(\begin{array}{cccc|c}
\frac{1}{3} & 1 & 0 & -\frac{1}{6} & \frac{1}{6} \\
0 & 0 & \frac{2}{3} & 1 & 1
\end{array}\right)
$$

$\frac{1}{6} R2 + R1 \to R1$

$$
\left(\begin{array}{cccc|c}
\frac{1}{3} & 1 & \frac{1}{9} & 0 & \frac{1}{3} \\
0 & 0 & \frac{2}{3} & 1 & 1
\end{array}\right)
$$

$\Rightarrow \mathcal{B} = \{2,4\}$, read off solution:

$$
\begin{array}{ccccc}
\text{put } -1 & & \text{put } -1 & & \\
\downarrow & & & & \\
\end{array}
$$

$$
\left(\begin{array}{cccc|c}
-1 & 0 & 0 & 0 & 0 \\
\frac{1}{3} & 1 & \frac{1}{9} & 0 & \frac{1}{3} \\
0 & 0 & -1 & 0 & 0 \\
0 & 0 & \frac{2}{3} & 1 & 1
\end{array}\right)
$$

$$
\Rightarrow \quad x = \begin{pmatrix} 0 \\ \frac{1}{3} \\ 0 \\ 1 \end{pmatrix} + \tilde{\tilde{\lambda}} \begin{pmatrix} -1 \\ \frac{1}{3} \\ 0 \\ 0 \end{pmatrix} + \tilde{\tilde{\mu}} \begin{pmatrix} 0 \\ \frac{1}{9} \\ -1 \\ \frac{2}{3} \end{pmatrix}
$$

Conclusion: We know how to compute all solutions to $Ax=b$ (A a wide matrix) in the specific form $x = x^{basic} + x^{hom}$, where x^{hom} solves $Ax^{hom}=0$, and x^{basic} has at least as many 0 entries as "number of columns" minus "number of pivots".