Operations Research, Constructor University Bremen, Fall 2023 Session 6

Prof. Sören Petrat Sep. 21, 2023

2.3 The Simplex Method

Conclusion from previous chapter:

- For a standard form LP problem we need to check only basic $\overbrace{\text{feasible}}^{\text{all components} \geq 0}$ solutions.
- We can find basic solutions with Gaussian elimination.

So in order to find optimal solutions, we could simply go through all possible basic feasible solution.

BUT: For large problems, this is way too computationally expensive!

The following method is faster:

Simplex algorithm:

(i) Start with any basic feasible solution.

(ii) Swap one basic variable ("leaving variable") for another variable ("entering variable") s.t. objective function improves the most.

(iii) Repeat this until no improvement of objective fct. is possible.

Let us work out the details using the example from Session 3:

$$\tilde{x} = \begin{pmatrix} x_1 \\ x_2 \\ u \\ v \\ s_1 \\ s_2 \\ s_3 \end{pmatrix}, \quad A = \begin{array}{c} \begin{array}{ccccccc} x_1 & x_2 & u & v & s_1 & s_2 & s_3 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \end{array} \\ \begin{pmatrix} 1 & 1 & -1 & 1 & 0 & 0 & 0 \\ 2 & -1 & -2 & 2 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 0 & 1 \end{pmatrix} \end{array}, \quad b = \begin{pmatrix} 1 \\ 5 \\ 4 \\ 5 \end{pmatrix}, \quad c = \begin{pmatrix} -1 \\ -2 \\ -3 \\ 3 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

We write it as a "simplex tableau":

$$\begin{array}{c|ccccccc|c}
 & x_1 & x_2 & u & v & s_1 & s_2 & s_3 & \\
 & 1 & 1 & -1 & 1 & 0 & 0 & 0 & 1 \\
A & 2 & -1 & -2 & 2 & 1 & 0 & 0 & 5 \quad b \\
 & 1 & -1 & 0 & 0 & 0 & 1 & 0 & 4 \\
 & 0 & 1 & 1 & -1 & 0 & 0 & 1 & 5 \\
\hline
c^T & -1 & -2 & -3 & 3 & 0 & 0 & 0 & z
\end{array}$$

$\leftarrow$ add $c^Tx = z$ as last row

(i) We need to find a basic solution. (Columns s_1, s_2, s_3 are already in the right form; we need to choose one more.)

Let us choose $x_1, \underbrace{s_1, s_2, s_3}$ columns as our pivot columns.

s_1, s_2, s_3 are already pivot columns.

We could have just as well chosen x_2, u, or v.

$\Rightarrow$ We need to eliminate ○ entries.

now these four columns are the pivot columns

$$\Rightarrow \quad \begin{array}{r|ccccccc|c}
 & x_1 & x_2 & u & v & s_1 & s_2 & s_3 & \\
 & 1 & 1 & -1 & 1 & 0 & 0 & 0 & 1 \\
-2R_1 + R_2 \to R_2: & 0 & -3 & 0 & 0 & 1 & 0 & 0 & 3 \\
-R_1 + R_3 \to R_3: & 0 & -2 & 1 & -1 & 0 & 1 & 0 & 3 \\
 & 0 & 1 & 1 & -1 & 0 & 0 & 1 & 5 \\
\hline
R_1 + R_5 \to R_5: & 0 & -1 & -4 & 4 & 0 & 0 & 0 & z+1
\end{array}$$

Note: By eliminating the "pivot column entry" here, we can read off z on the right ($z=-1$ here).

A basic $\underbrace{\text{feasible}}_{\text{meaning all components} \geq 0}$ solution is: $x_1 = 1, x_2 = 0, u = 0, v = 0, s_1 = 3, s_2 = 3, s_3 = 5.$

recall how we got basic solutions (sessions 4 and 5)

There, $\underset{\smile}{c^T}x = 0 = z+1$ i.e., $z = -1$.

c^T is the last row (left-hand side)

$$\tilde{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 3 \\ 3 \\ 5 \end{pmatrix}$$

Here, we are "lucky": Our basic solution is also feasible.

(If this were not the case, we have to try different pivot columns.)

Note: We make one more simplification in the notation:

$$\begin{array}{ccccccc|c} x_1 & x_2 & u & v & s_1 & s_2 & s_3 & \\ 1 & 1 & -1 & 1 & 0 & 0 & 0 & 1 \\ 0 & -3 & 0 & 0 & 1 & 0 & 0 & 3 \\ 0 & -2 & 1 & -1 & 0 & 1 & 0 & 3 \\ 0 & 1 & 1 & -1 & 0 & 0 & 1 & 5 \\ \hline 0 & -1 & -4 & 4 & 0 & 0 & 0 & 1 \end{array}$$

we delete the z here; then this number is equal to $-z$ at the basic solution.

this tracks $-z$, i.e., this shall be <u>maximized</u>

(ii) Entry variable: Go along direction that improves objective function the most.

new entry variable

$$\begin{array}{ccccccc|c} x_1 & x_2 & u & v & s_1 & s_2 & s_3 & \\ 1 & 1 & -1 & 1 & 0 & 0 & 0 & 1 \\ 0 & -3 & 0 & 0 & 1 & 0 & 0 & 3 \\ 0 & -2 & 1 & -1 & 0 & 1 & 0 & 3 \\ 0 & 1 & 1 & -1 & 0 & 0 & 1 & 5 \\ \hline 0 & -1 & -4 & 4 & 0 & 0 & 0 & 1 \end{array}$$

Geometrically: Go along direction where the slope is most negative.

choose the column variable where the entry in the last row is the most negative. (If all entries are positive, we are done, because z cannot be decreased further.)

Question: What is the leaving variable, i.e., in which row (in column u) should we put the new pivot?

At the basic solution, with including $u \geq 0$ (as the entering variable), we have:

needs to be $\geq$ to be feasible

$x_1 = 1 - (-1)u \geq 0 \Rightarrow$ no bound on u (any pos. u would work)

$s_1 = 3 - 0 \cdot u \geq 0 \Rightarrow$ no bound on u (any pos. u would work)

$s_2 = 3 - 1 \cdot u \geq 0 \Rightarrow$ need $u \leq \frac{3}{1}$ ← this is the strongest constraint

$s_3 = 5 - 1 \cdot u \geq 0 \Rightarrow$ need $u \leq \frac{5}{1}$ (e.g., if we would increase u up to 5, we would end up with an unfeasible solution)

How do x_1, s_1, s_2, s_3 change if we increase u until the respective variable (x_1, s_1, s_2, or s_3) is 0?

$\Rightarrow$ choose s_2

General rule: take the row with the least positive ratio of coefficient from right-most column to coefficient in new entry variable column.

$\Rightarrow$ Here, we need to take pivot in R_3, i.e., s_2 as leaving variable

We compute:

$$
\begin{array}{r|ccccccc|c}
 & x_1 & x_2 & u & v & s_1 & s_2 & s_3 & \\
\hline
R_3 + R_1 \to R_1: & 1 & -1 & 0 & 0 & 0 & 1 & 0 & 4 \\
 & 0 & -3 & 0 & 0 & 1 & 0 & 0 & 3 \\
 & 0 & -2 & 1 & -1 & 0 & 1 & 0 & 3 \\
R_4 - R_3 \to R_4: & 0 & 3 & 0 & 0 & 0 & -1 & 1 & 2 \\
\hline
4R_3 + R_5 \to R_5: & 0 & -9 & 0 & 0 & 0 & 4 & 0 & 13
\end{array}
$$

$\Rightarrow x_1 = 4, u = 3, s_1 = 3, s_3 = 2,$

$x_2 = 0, v = 0, s_2 = 0$

and $z = -13$

next entry variable: x_2

new pivot (only positive entry in this column) $\Rightarrow$ new leaving variable: s_3

$\Rightarrow$ We need to do another step.

(We are finished when all entries in the last row are non-negative!)