Another example for using Dynamic Programming: "Distribution of Effort" Problem

# of consulting teams	admin units 1	2	3
0	0	0	0
1	45	20	50
2	70	45	70
3	90	75	80
4	105	110	100
5	120	150	130
	$b_x^{(1)}$	$b_x^{(2)}$	$b_x^{(3)}$

← Amount of money saved (in 1000$)

(note: not proportional to # of teams!)

Goal: distribute 5 consulting teams to 3 units s.t. max. amount of money saved.

(E.g., 3 teams to unit 1 saves 90, 1 team to unit 2 saves 20, 1 team to unit 3 saves 50

=> total saving of 160)

Note: This problem type is called "distribution of effort problem".

Dynamic Programming formulation:

- stages = units, x_i = # of teams sent to unit i, s = # of teams still available
- here: $f(s,x_i) = \underbrace{b^{(i)}_{x_i}}_{\text{see table above}} + f^*_{i+1}(\underbrace{s-x_i}_{\text{\# of teams available in next step (the new } s \text{ in the next step)}})$

Solution:

$i=3$:

unit 3
(could have started with any unit here)

s	f_3^*	x_3^*
0	0	0
1	50	1
2	70	2
3	80	3
4	100	4
5	130	5

$i=2$:

s	$f_2(s,x_2)=b^{(2)}_{x_2}+f_3^*(s-x_2)$ $x_2=0$	$x_2=1$	$x_2=2$	$x_2=3$	$x_4=4$	$x_2=5$	$f_2^*(s)$	x_2^*
0	0+0	–	–	–	–	–	0	0
1	0+50	20+0	–	–	–	–	50	0
2	0+70	20+50	45+0	–	–	–	70	0 or 1
3	0+80	20+70	45+50	75+0	–	–	95	2
4	0+100	20+80	45+70	75+50	110+0	–	125	3
5	0+130	20+100	45+80	75+70	110+50	150+0	160	4

$i=1$:

s here

s	$f_1(s,x_1)=b^{(1)}_{x_1}+f_2^*(5-x_1)$ $x_1=0$	$x_1=1$	$x_1=2$	$x_1=3$	$x_1=4$	$x_1=5$	$f_1^*(s)$	x_1^*
5	0+160 =160	45+125 =170	70+95 =165	90+70 =160	105+50 =155	120+0 =120	170	1

$\Rightarrow$ Solution: send 1 team to unit 1 (4 remaining), 3 to unit 2 (1 remaining), and 1 to unit 3, to save the maximal 170 (thousand \$).

Another example involving continuous optimization: Local job shop problem (Hillier, Lieberman: Chapter 11.3)

Setup: different seasons with minimal worker requirements:

Season	Spring	Summer	Autumn	Winter	Spring	...
Requirement	255	220	240	200	255	...

Too much employment costs 2000\$ per person per season.

Changing employment from one season to the next costs 200\$ $\cdot$ (difference in employment)^2.

We assume fractional employment is possible (part-time work). (I.e., optimal solution need not be integer.)

Goal: Find hiring schedule that minimizes costs.

We introduce: • Stage 4 = spring, Stage 3 = winter, Stage 2 = autumn, Stage 1 = summer

highest employment level

• x_n = employment level for stage n. Note: $x_4 = 255$.

highest number

• r_n = minimum requirements from table above.

=> Feasible values for x_n: $r_n \leq x_n \leq 255$

Note: The states $s_n = x_{n-1}$ can now take a continuum of values.

Similar to before, we set

$$f_n(s_n, x_n) = \underbrace{200(x_n - s_n)^2}_{\text{employment change cost}} + \underbrace{2000(x_n - r_n)}_{\text{extra employment cost}} + \underbrace{f^*_{n+1}(x_n)}_{\text{optimal costs for later stages}}$$

} cost given employment s_n at stage $n-1$, and x_n at stage n, and optimal future cost

$$f^*_n(s_n) = \min_{r_n \leq x_n \leq 255} f_n(s_n, x_n)$$

} optimal cost given employment s_n at stage $n-1$, and optimal in the future

Solution:

Stage $n = 4$:

s_4	$f^*_4(s_4)$	x^*_4
$200 \leq s_4 \leq 255$	$200(255 - s_4)^2$	255

We continue next time...