3.3 Inventory Theory

Inventory management is very important in the business world e.g., for
- retail
- factories (materials for production / resources need to be available)

General ideas:
- Costs for storing ("carrying") and resupplying inventory, but also penalties if not enough inventory available.
- First, we look at deterministic models, where the demand is known (e.g., production). After, we look at stochastic models, where demand is a random variable.

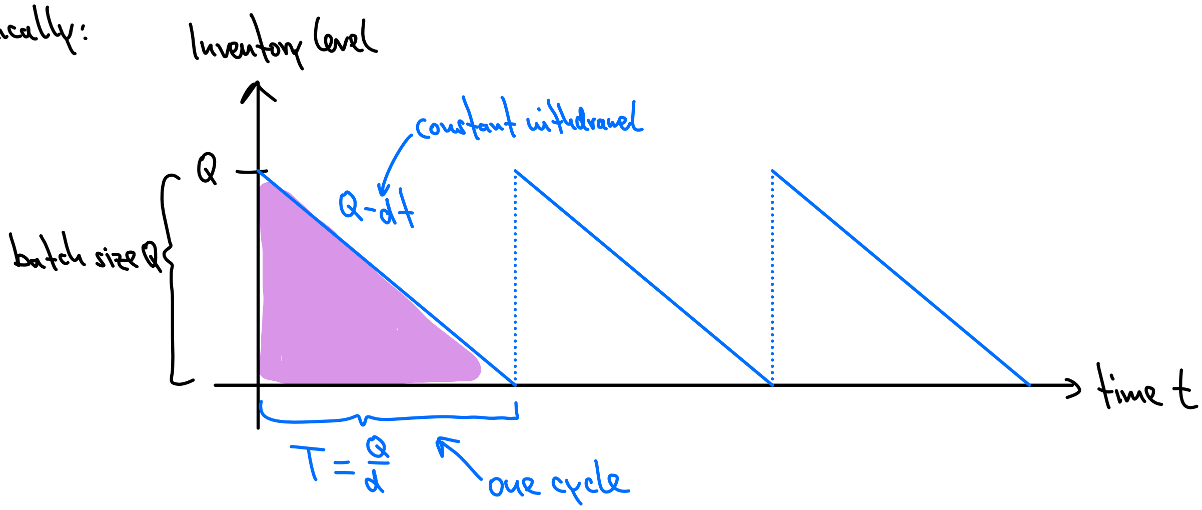
First, let us consider inventory management under the following assumptions:

- The cost of ordering is:
  - K setup costs per order,
  - c unit costs.
- The holding (or storage) cost is h per unit per time in inventory.
- There is a constant withdrawal rate of d units per time.
- We do not allow for shortages.
- There is continuous review, i.e., inventory level is continuously checked (as opposed to periodic checks)

These assumptions lead to the basic $\boxed{\text{economic order quantity model (EOQ)}}$.

Note: Under these assumptions, it is always optimal that new orders arrive exactly when inventory is empty.

Graphically:



The cost per cycle is then

$$C_{cycle} = \underbrace{K + cQ}_{\text{order costs}} + \underbrace{h \frac{Q}{2} T}_{\text{holding costs}}$$

holding costs = $h \cdot$ (area of triangle)

$$= h \int_0^T (Q - td)\, dt = h\left(QT - \frac{T^2}{2} d\right) = h \frac{QT}{2}$$

$$\Rightarrow \text{Total cost per time } C = \frac{C_{cycle}}{T} = \frac{K + cQ + h\frac{Q}{2}T}{T} = \frac{K+cQ}{T} + h\frac{Q}{2} = \frac{K+cQ}{Q/d} + h\frac{Q}{2}$$

$$\Rightarrow C = \frac{dK}{Q} + dc + h\frac{Q}{2}$$

What is the optimal order quantity Q^* that minimizes cost per time C?

→ We need to find the minimum:

$$\frac{dC}{dQ} = -\frac{dK}{Q^2} + \frac{h}{2} \stackrel{!}{=} 0 \quad \Rightarrow \quad \boxed{Q^* = \sqrt{\frac{2dK}{h}} \qquad \text{(EOQ formula)}}$$

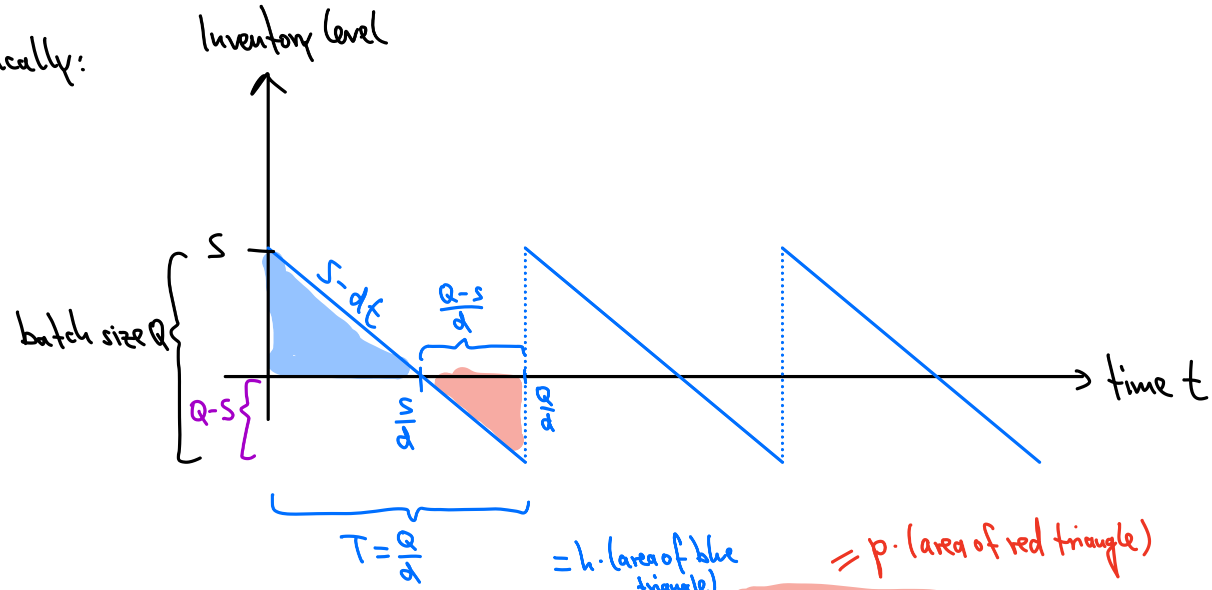
The corresponding optimal cycle time is $\boxed{T^* = \frac{Q^*}{d} = \sqrt{\frac{2K}{dh}}}$

Next:

Let us assume the inventory can be empty for part of a cycle at a penalty p per unit per time. Withdrawals that cannot be fulfilled will be postponed and processed when new batch arrives.

This leads to the EOQ model with planned shortages

Graphically:



$$\Rightarrow \text{Cycle cost } C_{\text{cycle}} = \underbrace{K + cQ}_{\text{order cost}} + \underbrace{\boxed{\overbrace{h\,\frac{S}{2}\,\frac{S}{d}}^{=h\cdot(\text{area of blue triangle})}}}_{\text{holding cost}} + \underbrace{\boxed{\overbrace{p\,\frac{Q-S}{2}\,\frac{Q-S}{d}}^{=p\cdot(\text{area of red triangle})}}}_{\text{penalty}}$$

$$\Rightarrow \text{Total cost per time } C = \frac{C_{\text{cycle}}}{T} = \frac{K}{Q/d} + \frac{cQ}{Q/d} + \frac{1}{2}\,\frac{hS^2}{d\,Q/d} + \frac{1}{2}\,p\,\frac{(Q-S)^2}{d\,Q/d}$$

$$= \frac{dK}{Q} + dc + \frac{1}{2}\,\frac{hS^2}{Q} + \frac{1}{2}\,p\,\underbrace{\frac{(Q-S)^2}{Q}}_{=\frac{(Q-S)}{Q}(Q-S) = \left(1-\frac{S}{Q}\right)(Q-S)}$$

Here, Q and S are decision variables, so we need to compute two partial derivatives:

$$\frac{\partial C}{\partial S} = \frac{hS}{Q} - p\,\frac{(Q-S)}{Q} \overset{!}{=} 0 \quad \Rightarrow \underbrace{hS = p(Q-S)}_{(*)} \quad \Rightarrow (h+p)\,S = pQ$$

$$\Rightarrow S = \frac{p}{h+p}\,Q$$

$$\frac{\partial C}{\partial Q} = -\frac{dK}{Q^2} - \frac{1}{2}\frac{hS^2}{Q^2} + \frac{1}{2}p\left(\frac{S}{Q^2}(Q-S) + \underbrace{1-\frac{S}{Q}}_{=\frac{Q-S}{Q}}\right)$$

$$= -\frac{dK}{Q^2} - \frac{1}{2}\frac{hS^2}{Q^2} + \frac{1}{2}\underbrace{p(Q-S)}_{=hS \text{ (see Equation (*))}}\left(\frac{S}{Q^2}+\frac{1}{Q}\right) \overset{!}{=} 0 \qquad (*)$$

$$\Longrightarrow (*) \Longrightarrow -\frac{dK}{Q^2} - \frac{1}{2}\frac{hS^2}{Q^2} + \frac{1}{2}hS\left(\frac{S}{Q^2}+\frac{1}{Q}\right) = 0$$

$$\Longleftrightarrow -\frac{dK}{Q^2} + \frac{1}{2}\frac{hS}{Q} = 0$$

$S = \frac{p}{h+p}Q$

$$\Longrightarrow \frac{dK}{Q^2} = \frac{1}{2}\frac{hp}{h+p}$$

$$\Longrightarrow \boxed{Q^* = \underbrace{\sqrt{\frac{2dK}{h}}}_{\text{known part from previous EOQ formula}}\sqrt{\frac{h+p}{p}}} \text{ is the minimum}$$

with corresponding $S^* = \frac{p}{h+p}Q^* = \sqrt{\frac{2dK}{h}}\sqrt{\frac{p}{h+p}}$,

and cycle time $T^* = \frac{Q^*}{d} = \sqrt{\frac{2K}{dh}}\sqrt{\frac{p}{h+p}}$.

Note: If $\underbrace{p \to \infty}_{\text{very high penalty, so no shortage should be optimal}}$, then $\sqrt{\frac{h+p}{p}} \to 1$, and we recover the basic EOQ model from before.