

# Advanced Calculus and Methods of Mathematical Physics

## Homework 3

Due on February 28, 2023, before the tutorial.

### Problem 1 [4 points]

Prove the Cauchy–Schwarz inequality:

$$|\langle x, y \rangle| \leq \|x\| \|y\|$$

for all vectors  $x, y$  in an inner product space. (*Hint: One possibility is to start with computing  $\langle x + \lambda y, x + \lambda y \rangle$  for any  $\lambda \in \mathbb{C}$  using the properties of the inner product. Then a specific choice for  $\lambda$  yields the inequality.*)

### Problem 2 [6 points]

The operator norm for linear maps  $A$  between normed vector spaces  $X$  and  $Y$  is defined by

$$\|A\| := \sup_{x \in X, \|x\|=1} \|Ax\|.$$

- (a) Verify that the operator norm is indeed a norm.
- (b) Let  $X, Y$ , and  $Z$  be normed vector spaces, and let  $B: X \rightarrow Y$  and  $A: Y \rightarrow Z$  be linear maps. Show that

$$\|AB\| \leq \|A\| \|B\|.$$

### Problem 3 [4 points]

Draw the following unit balls in a two-dimensional coordinate system:

- (a)  $B_1 := \{y \in \mathbb{R}^2 : \|y\|_1 \leq 1\}$ ,
- (b)  $B_2 := \{y \in \mathbb{R}^2 : \|y\|_2 \leq 1\}$ ,
- (c)  $B_3 := \{y \in \mathbb{R}^2 : \|y\|_3 \leq 1\}$ ,
- (d)  $B_\infty := \{y \in \mathbb{R}^2 : \|y\|_\infty \leq 1\}$ .

Recall that

$$\|y\|_p := \left( \sum_i |x_i|^p \right)^{1/p}.$$

**Problem 4 [6 points]**

We denote by  $X$  the set of all real-valued continuous functions on the interval  $[a, b]$ . For  $f, g \in X$ , we define

$$d_1(f, g) := \int_a^b |f(x) - g(x)| dx, \quad d_\infty(f, g) := \max_{x \in [a, b]} |f(x) - g(x)|.$$

- (a) Show that  $(X, d_1)$  and  $(X, d_\infty)$  are metric spaces.
- (b) Show that if  $f_n \rightarrow f$  in  $(X, d_\infty)$ , then also  $f_n \rightarrow f$  in  $(X, d_1)$ .
- (c) Show that there is a sequence  $(f_n)$  in  $X$  that converges to some  $f$  in  $(X, d_1)$  but that does not converge to  $f$  in  $(X, d_\infty)$ .