A few more remarks about power series:

We define the $\boxed{\text{Cauchy product}}$ of $\sum a_k$ and $\sum b_k$ by $\sum c_k$ with $c_k = \sum_{\ell=0}^{k} a_\ell b_{k-\ell}$.

$\left((a_0+a_1+a_2+\dots)(b_0+b_1+b_2+\dots) = a_0b_0 + (a_1b_0+a_0b_1) + (a_2b_0+a_1b_1+a_0b_2) + \dots \right)$

Result: If $\sum a_k$ conv. to A and $\sum b_k$ conv. to B, and at least one of the series conv. absolutely, then $\sum_{k=0}^{\infty}\sum_{\ell=0}^{k} a_\ell b_{k-\ell}$ conv. absolutely to AB.

Next, let us consider a (complex) $\boxed{\text{power series}}$ $\underbrace{\sum_{k=0}^{\infty} c_k z^k}$, with $c_k, z \in \mathbb{C}$.

more generally, we could consider $\sum_{k=0}^{\infty} c_k (z-c)^k$

From the root test we get that it converges if

$$\limsup_{k\to\infty} \sqrt[k]{|c_k||z|^k} = \limsup_{k\to\infty} \sqrt[k]{|c_k|}\,|z| < 1 \quad \text{i.e., if } |z| < \boxed{\frac{1}{\limsup_{k\to\infty} \sqrt[k]{|c_k|}} =: \rho}.$$

This ρ is called $\boxed{\text{radius of convergence}}$ (possibly 0 or ∞).

Alternatively: Whenever ratio test applies, then $\rho = \lim_{k\to\infty} \left|\frac{c_k}{c_{k+1}}\right|$.

So for all $|z|<\rho$ (i.e., on an open disc with radius ρ), $\sum c_k z^k$ conv. absolutely, and for $|z|>\rho$ it diverges. (For $|z|=\rho$ it might or might not converge.)

In fact: $\sum c_k z^k$ converges uniformly on the set $\{z\in\mathbb{C} : |z|\le \rho-\varepsilon\}$ for any $\varepsilon>0$.

For a real power series $f(x) = \sum a_k x^k$, the previous thm. says:

- f is differentiable on $(-\varrho, \varrho)$ and $f'(x) = \sum k a_k x^{k-1}$,
- for any $[0,x] \subset (-\varrho,\varrho)$ we have $\int_0^x f(t)\,dt = \sum_{k=0}^{\infty} \frac{a_k}{k+1} x^{k+1} + \text{const.}$

we can take derivatives and integrate term-wise

Ex.: $\sum_k \frac{x^k}{k!}$ has radius of convergence $\varrho = \lim_{k\to\infty} \frac{1/k!}{1/(k+1)!} = \lim_{k\to\infty}(k+1) = \infty$, so it converges $\forall x \in \mathbb{R}$, and $\frac{d}{dx} \sum_{k=0}^{\infty} \frac{x^k}{k!} = \sum_{k=0}^{\infty} k \frac{x^{k-1}}{k!} = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ (i.e., $\frac{d}{dx} e^x = e^x$).