Advanced Calculus and Methods of Mathematical Physics — Session 9

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Recall:

- Differentiability: $f(\tilde{x}+h) = f(\tilde{x}) + \underbrace{Df|_{\tilde{x}}}_{\text{total derivative; unique}} h + r_{\tilde{x}}(h)$ with $\lim_{h\to 0} \frac{\|r_{\tilde{x}}(h)\|}{\|h\|} = 0$.

- Directional derivative: $D_u f|_{\tilde{x}} = \lim_{t\to 0} \frac{f(\tilde{x}+tu) - f(\tilde{x})}{t}$ $\quad (u \in \mathbb{R}^n, \|u\| = 1)$.

- Partial derivatives $D_{e_j} f|_{\tilde{x}} = \frac{\partial f}{\partial x_j}(\tilde{x})$.

We proved: Differentiability $\Longrightarrow$ All directional derivatives exist and

$$D_u f|_{\tilde{x}} = Df|_{\tilde{x}}\, u. \leftarrow \text{linear in } u \text{ (see also HW)}$$

(in particular: $\frac{\partial f_i}{\partial x_j}(\tilde{x}) = \left(Df|_{\tilde{x}}\right)_{ij}$ ← Jacobian matrix)

Two examples (see also geogebra visualizations):

- $f(x,y) = \begin{cases} \frac{2xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$ Here, the partial derivatives exist at $(0,0)$, but f is not even continuous there.

- $f(x,y) = \begin{cases} \frac{xy^2}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$ Here, f is continuous at $(0,0)$ and all directional derivatives exist there. But f is not differentiable at $(0,0)$. (Geometrically: cannot put a tangent plane at origin.)

We need continuity of the partial derivatives to conclude (continuous) differentiability.

Theorem: Let $f: U \to \mathbb{R}^m$, $U \subset \mathbb{R}^n$ open. Then f is totally continuously differentiable on U if and only if all partial derivatives exist and are continuous on U.

Proof:

"$\Rightarrow$" We assume f is differentiable with $Df|_x$ continuous. Then from the previous theorem we know: $(Df|_x)_{ij} := \langle e_i, Df|_x e_j \rangle = \frac{\partial f_i}{\partial x_j}$ $\forall i,j$ and $x \in U$.

Thus: $$\left| \frac{\partial f_i}{\partial x_j}(x) - \frac{\partial f_i}{\partial x_j}(y) \right| = \langle e_i, (Df|_x - Df|_y) e_j \rangle$$

$$\leq \| Df|_x - Df|_y \| \quad \text{by Cauchy-Schwarz (and } \|e_i\| = 1)$$

$\Rightarrow \frac{\partial f}{\partial x_j}$ continuous. (If $\|x-y\| < \delta \Rightarrow \| Df|_x - Df|_y \| < \varepsilon$ (cont. of Df) and thus also $\left| \frac{\partial f_i}{\partial x_j}(x) - \frac{\partial f_i}{\partial x_j}(y) \right| < \varepsilon$.)

"$\Leftarrow$" We assume $\frac{\partial f_i}{\partial x_j}$ exists and is continuous on U $\forall i,j$.

We will show that this implies that f is differentiable. Then we know $(Df)_{ij} = \frac{\partial f_i}{\partial x_j}$ and continuity of $(Df)_{ij}$ implies continuity of Df.

Fix a component $f_i: U \to \mathbb{R}^m$, $x \in U$, $\varepsilon > 0$.

Then $\frac{\partial f_i}{\partial x_j}$ continuous $\Rightarrow \exists\, \delta > 0$ s.t. $y \in B_\delta(x) \subset U$ and $\left| \frac{\partial f_i}{\partial x_j}(x) - \frac{\partial f_i}{\partial x_j}(y) \right| < \frac{\varepsilon}{nm}$ $\forall j = 1, \dots, n$.

Now let $h \in \mathbb{R}^n$, $\|h\| < \delta$, $h = \sum_{j=1}^n h_j e_j$.

Define $v_0 = 0$, $v_1 = h_1 e_1$, $v_2 = h_1 e_1 + h_2 e_2, \dots$, i.e., $v_k = \sum_{j=1}^k h_j e_j = \begin{pmatrix} h_1 \\ \vdots \\ h_k \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

Then $f_i(x+h) - f_i(x) = f_i(x+v_n) - f_i(x+v_0) \underset{\text{"telescope sum"}}{=} \sum_{j=1}^n \left(f_i(x+v_j) - f_i(x+v_{j-1}) \right)$.

For each summand we now use the 1-dimensional mean-value theorem:

$$f_i(x+v_j) - f_i(x+v_{j-1}) = f_i(x+v_{j-1}+h_je_j) - f_i(x+v_{j-1})$$

$$= h_j \frac{\partial f_i}{\partial x_j}\big(\underbrace{x+v_{j-1}+c_jh_je_j}_{\in B_\delta(x)}\big) \text{ for some } c_j \in (0,1).$$

$$\Rightarrow \left| f_i(x+v_j) - f_i(x+v_{j-1}) - h_j \frac{\partial f_i}{\partial x_j}(x) \right|$$

by continuity of $\frac{\partial f_i}{\partial x_j}$

$$= |h_j| \left| \frac{\partial f_i}{\partial x_j}(x+v_{j-1}+c_jh_je_j) - \frac{\partial f_i}{\partial x_j}(x) \right| < |h_j| \frac{\varepsilon}{nm}.$$

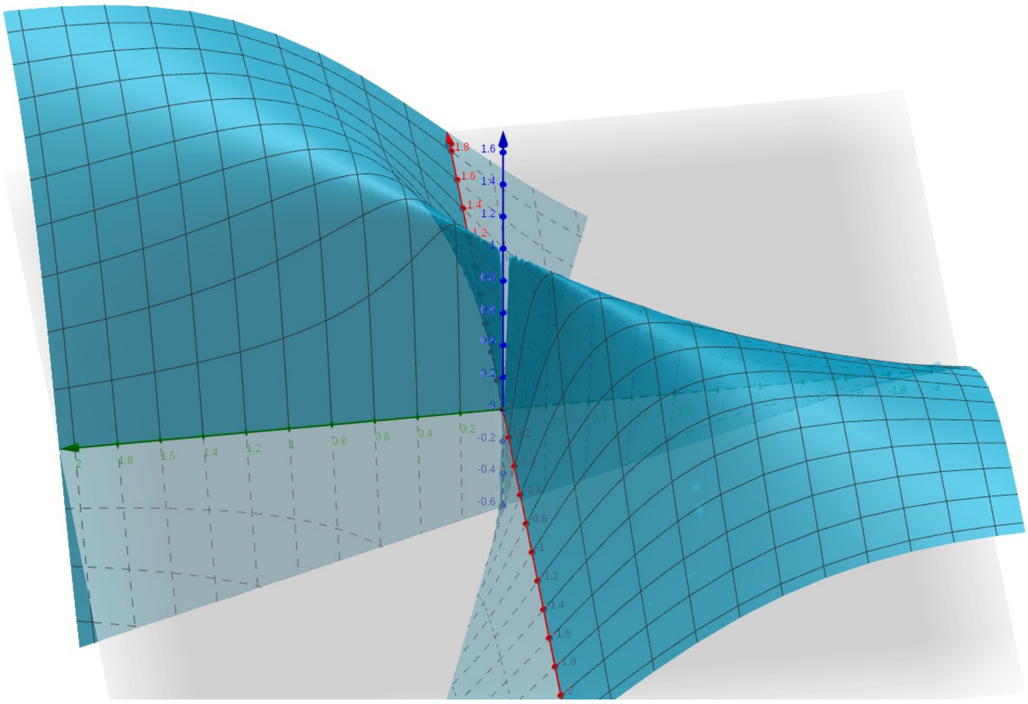
Together: $\left| f_i(x+h) - f_i(x) - \sum_{j=1}^{n} h_j \frac{\partial f_i}{\partial x_j}(x) \right| \leq \sum_{j=1}^{n} |h_j| \frac{\varepsilon}{nm} \leq \frac{\|h\|}{m} \varepsilon.$

In total: $\left\| f(x+h) - f(x) - \underbrace{\sum_{j=1}^{n} (D_{e_j} f) h_j}_{= Df|_x h} \right\| \leq \sum_{i=1}^{m} \frac{\|h\|}{m} \varepsilon = \|h\| \varepsilon$

$\Rightarrow \quad \frac{\|r_x(h)\|}{\|h\|} < \varepsilon \quad (\forall \varepsilon > 0) \Rightarrow f$ is differentiable at x. $\square$

See https://www.geogebra.org/3d for the plots.

$$f(x,y)=\frac{2xy}{x^2+y^2} \quad \text{for} \quad (x,y)\neq(0,0) \quad \text{and} \quad f(0,0)=(0,0)$$



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