Advanced Calculus and Methods of Mathematical Physics — Session 13

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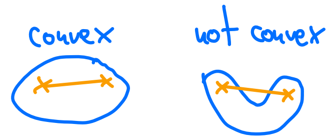
Today, we prove parts of the Inverse and Implicit Function Theorems.

We will need the following lemma:

All points on a straight line between any $x, y \in U$ are in U.



Lemma: Let $f: U \to \mathbb{R}^n$, $U \subset \mathbb{R}^n$ open and convex. If f is differentiable and $\exists M > 0$ s.t. $\|f'(x)\| \leq M$ $\forall x \in U$, then $\|f(x_1) - f(x_2)\| \leq M \|x_1 - x_2\|$ $\forall x_1, x_2 \in U$.

Proof: Define curve $\gamma: [0,1] \to \mathbb{R}^n$, $t \mapsto t x_1 + (1-t) x_2$. Then γ is in U because U is convex. If $g(t) := f(\gamma(t))$, then

$$f(x_1) - f(x_2) = g(1) - g(0) = \int_0^1 g'(t)\,dt, \text{ where } g'(t) \overset{\text{chain rule}}{=} f'(\gamma(t))\,\gamma'(t) = f'(\gamma(t))\,(x_1 - x_2).$$

$$\Rightarrow \|f(x_1) - f(x_2)\| \leq \int_0^1 \|g'(t)\|\,dt = \int_0^1 \underbrace{\|f'(\gamma(t))\|}_{\leq M} \|x_1 - x_2\|\,dt \leq M \|x_1 - x_2\|. \qquad \square$$

Proof of Inverse Function Theorem:

a) Ideas:
- Injectivity: We conclude $f(x) = y$ for at most one x by constructing a contraction s.t. x = fixed point (if it exists, it is unique).
- Surjectivity: We construct a complete metric space X s.t. we can apply the Banach fixed point theorem.

- Injectivity:

Let us call $A := Df|_p$, and define a constant $\lambda := \dfrac{1}{2\|A^{-1}\|}$.

Since Df is continuous at p, $\exists$ open ball $V \subset U$ centered at p s.t.

$$\|Df|_x - Df|_p\| \leq \lambda \quad \forall x \in V.$$

For any (fixed) $y \in \mathbb{R}^n$, we def. $\varphi_y(x) := x + A^{-1}(y - f(x))$, because then:

$$f(x) = y \iff \varphi_y(x) = x \quad (x \text{ a fixed point})$$

Is φ_y a contraction? If yes, then $\exists$ at most one fixed point (as we showed last time), i.e., $f(x) = y$ for at most one x, i.e., $f|_V$ is injective.

We try to bound $\|\varphi_y(x_1) - \varphi_y(x_2)\|$ by bounding the derivative $D\varphi_y|_x$ because of our lemma above.

We compute: $D\varphi_y|_x = 1 - A^{-1} Df|_x = A^{-1}(A - Df|_x)$.

Then for $x \in \tilde{U}$ we have $\|D\varphi_y|_x\| \leq \|A^{-1}(A - Df|_x)\|$

$$\leq \|A^{-1}\| \underbrace{\|Df|_p - Df|_x\|}_{\leq \lambda = \frac{1}{2\|A^{-1}\|}}$$

$$\leq \frac{1}{2}$$

Thus, by the lemma, we have $\|\varphi_y(x_1) - \varphi_y(x_2)\| \leq \frac{1}{2}\|x_1 - x_2\| \quad \forall x_1, x_2 \in V$, i.e., φ_y is a contraction.

- Surjectivity: We now know $f|_V : V \to f(V) =: W$ is injective, and thus surjective onto its image. But is W open?

  We pick $y_0 \in W$, so there is $x_0 \in V$ s.t. $f(x_0) = y_0$. Choose $\overline{B}$ the closed ball of radius r around x_0, with r so small that $\overline{B} \subset V$. We now try to show that $y \in V$ if $\|y - y_0\| < \lambda r$, implying that W is open. (W is open if $\forall y_0 \in W \; \exists B_r$ s.t. $B_r \subset W$).

To do this we aim to prove that $\varphi_y : \overline{B} \to \overline{B}$ is a contraction. Then, Banach's Fixed Point Theorem tells us that there is a fixed point, since $\overline{B}$ is complete.

Thus, $\varphi_y(x) = x \Rightarrow f(x) = y$, with $y \in f(\overline{B}) \subset f(V) = W$.

Left to show: φ_y maps into $\overline{B}$. So choose $x \in \overline{B}$ and try to prove $\|\varphi_y(x) - x_0\| \leq r$.

We check: $\|\varphi_y(x) - x_0\| \overset{\text{triangle ineq.}}{\leq} \underbrace{\|\varphi_y(x) - \varphi_y(x_0)\|}_{\leq \frac{1}{2}\|x - x_0\| \leq \frac{r}{2} \text{ (from above)}} + \|\varphi_y(x_0) - x_0\| \leq r \quad \checkmark$

$$\|\varphi_y(x_0) - x_0\| = \|x_0 + A^{-1}(y - f(x_0)) - x_0\|$$
$$= \|A^{-1}(y - y_0)\|$$
$$\leq \underbrace{\|A\|^{-1}}_{\leq \frac{1}{2\lambda}} \underbrace{\|y - y_0\|}_{\leq \lambda r} \leq \frac{r}{2}$$

b) First, we use: If $Df|_p$ has an inverse, then also $Df|_x$ for $\|x - p\|$ small enough has an inverse. (The set of invertible linear maps on $\mathbb{R}^n$ is open; see Rudin for a proof.)

Then differentiability of f^{-1} can be proven by showing

$$\frac{f^{-1}|_{y+k} - f^{-1}_y - (Df|_x)^{-1} k}{\|k\|} \xrightarrow{k \to 0} 0. \quad \text{(See Rudin for the details.)}$$

Continuity of Df^{-1} follows from continuity of $Df|_x$. $\square$

Idea of proof of Implicit Function Theorem:

We define $F: U \to \mathbb{R}^{n+m}$ by $F(x,y) := \begin{pmatrix} f(x,y) \\ y \end{pmatrix}$.

Then $F(p,q) = (0,q)$ and $DF|_{(p,q)} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \underbrace{0}_{\frac{\partial y}{\partial x}} & \underbrace{\mathbb{1}}_{\frac{\partial y}{\partial y}} \end{pmatrix}$.

By assumption $\frac{\partial f}{\partial x}$ is invertible, i.e., all columns are linearly independent. But then due to the $\mathbb{1}$, all columns of $DF|_{(p,q)}$ are linearly indep. as well, so $DF|_{(p,q)}$ is invertible and we can use the inverse fct. thm. to invert F in neighborhoods V of (p,q) and W of $(0,q)$.

If $(0,y) \in W$, then $(0,y) = F(x,y)$ for some $(x,y) \in V$, i.e., $f(x,y) = 0$.

Next:
- x is unique since F is bijective.
- The fct. g is C^1 since F^{-1} is C^1.
- Derivative formula follows from chain rule.

} see Rudin for details

$\square$

Note: In our chapter on many variable differentiation, we skipped a discussion on Lagrange multipliers. This is discussed in Calculus and Linear Algebra II.