Advanced Calculus and Methods of Mathematical Physics — Session 15

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Let us discuss one example of applying the Leibniz rule: $\left(F(y) := \int_0^\infty f(x,y)\, dx \right)$

<u>Example:</u> We aim at computing $\int_0^\infty \frac{\sin x}{x} dx$. For that we use the following trick:

We def. $f(x,y) = e^{-xy} \frac{\sin x}{x}$ on $I = [0,\infty) \times [\alpha, \beta]$, $0 < \alpha < \beta$ (and $f(0,y) := 1$).

Then $\frac{\partial f}{\partial y}(x,y) = -e^{-xy} \sin x$ (which we know how to integrate).

We have $y \geq \alpha > 0$ (so $e^{-xy} \leq e^{-x\alpha}$) and $\left|\frac{\sin x}{x}\right| \leq 1$ (and $|\sin x| \leq 1$), so conditions (i) and (ii) from Leibniz rule II hold.

We find
$$F'(y) = \int_0^\infty \frac{\partial f}{\partial y}(x,y)\, dx = -\int_0^\infty e^{-xy} \sin x\, dx$$

integration by parts

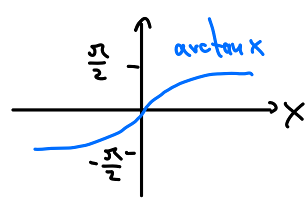
$$= \underbrace{-e^{-xy}(-\cos x)\Big|_0^\infty}_{= 0 - 1} + (-y) \int_0^\infty e^{-xy}(-\cos x)\, dx$$

$$= -1 + y \left[\underbrace{e^{-xy} \sin x \Big|_0^\infty}_{= 0 - 0} - (-y) \int_0^\infty e^{-xy} \sin x\, dx \right]$$

$$= -1 + y^2 \underbrace{\int_0^\infty e^{-xy} \sin x\, dx}_{= -F'(y)}$$

$$\Rightarrow F'(y) = -1 - y^2 F'(y) \Rightarrow F'(y) = \frac{-1}{1+y^2}$$

$$\Rightarrow F(\beta) - F(\alpha) = \int_\alpha^\beta F'(y)\, dy = \arctan\alpha - \arctan\beta.$$



Note that $|F(\beta)| \leq \int_0^\infty e^{-x\beta} dx = -\frac{1}{\beta} e^{-\beta x}\Big|_0^\infty = \frac{1}{\beta} \xrightarrow{\beta \to \infty} 0$, so for $\beta \to \infty$ we get

$$0 - F(\alpha) = \arctan\alpha - \frac{\pi}{2} \Rightarrow F(\alpha) = \frac{\pi}{2} - \arctan\alpha, \text{ where } F(\alpha) = \int_0^\infty e^{-x\alpha} \frac{\sin x}{x}\, dx.$$

It can be shown that F is indeed continuous (from the right), so we have computed the Dirichlet integral $F(0) = \int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$. ("Feynman's trick")

For the next generalization of the Leibniz rule, the integration boundaries can be variable:

Theorem (Leibniz rule III): For $I = [a,b] \times [\alpha,\beta] = I_1 \times I_2$, let f and $\frac{\partial f}{\partial y} \in C(I)$, and $\phi, \psi : I_2 \to I_1$ be $C^1(I_2)$. Let

$$H(y) := \int_{\phi(y)}^{\psi(y)} f(x,y)\,dx.$$

Then $H \in C^1(I_2)$ and $H'(y) = \int_{\phi(y)}^{\psi(y)} \frac{\partial f}{\partial y}(x,y)\,dx + f(\psi(y),y)\,\psi'(y) - f(\phi(y),y)\,\phi'(y)$.

Proof: Define $F(y,u,v) := \int_u^v f(x,y)\,dx$ and $G(y) = (y, \phi(y), \psi(y))$, then $\underbrace{H = F \circ G}_{H(y) = F(G(y))}$.

For fixed u and v, F satisfies the conditions of Leibniz rule I, so $\frac{\partial F}{\partial y} = \int_u^v \frac{\partial f}{\partial y}(x,y)\,dx$.

Then the chain rule gives: $H'(y) = (dF \circ G)\,G' = \left(\int_u^v \frac{\partial f}{\partial y}(x,y)\,dx, -f, f \right) \circ G \begin{pmatrix} 1 \\ \phi'(y) \\ \psi'(y) \end{pmatrix}$

$$= \int_{\phi(y)}^{\psi(y)} \frac{\partial f}{\partial y}(x,y)\,dx - f(\phi(y),y)\,\phi'(y) + f(\psi(y),y)\,\psi'(y). \quad \square$$

With these three theorems, we have a good understanding of how to exchange integration and differentiation. (Note: Much nicer conditions hold for the Lebesgue integral → Analysis III.)

Next: Does the order of integration matter?

Theorem: Let $f \in C(I)$, $I = [a,b] \times [\alpha, \beta]$. Then

$$\int_\alpha^\beta \int_a^b f(x,y)\,dx\,dy = \int_a^b \int_\alpha^\beta f(x,y)\,dy\,dx.$$

Proof: Idea: estimate integrals on small rectangles and then use uniform continuity.

Let $\varepsilon > 0$. Since f uniformly continuous:

$$\exists \delta > 0 \text{ s.t. if } d((x,y),(x',y')) < \delta \text{, then } |f(x,y) - f(x',y')| < \frac{\varepsilon}{(b-a)(\beta-\alpha)}. \quad (*)$$

Now we partition the x and y axis: Def. $a = x_0 < x_1 < \dots < x_n = b$ and $\alpha = y_0 < y_1 < \dots < y_m = \beta$ such that $I_{ij} := [x_{i-1}, x_i] \times [y_{j-1}, y_j]$ has diameter smaller δ $\forall i,j$.

We def. $m_{ij} = \min_{(x,y) \in I_{ij}} f(x,y)$ and $M_{ij} = \max_{(x,y) \in I_{ij}} f(x,y)$.

If $A_{ij} = \text{area}(I_{ij}) = (x_i - x_{i-1})(y_j - y_{j-1})$, then

$$m_{ij} A_{ij} \le \int_{x_{i-1}}^{x_i} \int_{y_{j-1}}^{y_j} f(x,y)\,dy\,dx \le M_{ij} A_{ij}.$$

Summing up yields:

$$\sum_{i,j} m_{ij} A_{ij} \le \int_a^b \int_\alpha^\beta f(x,y)\,dy\,dx \le \sum_{i,j} M_{ij} A_{ij}. \quad (**)$$

This argument works just as well for the other order of integration, i.e.,

$$\sum_{i,j} m_{ij} A_{ij} \le \int_\alpha^\beta \int_a^b f(x,y)\,dx\,dy \le \sum_{i,j} M_{ij} A_{ij}. \quad (**)'$$

Since $\left| \sum_{i,j} m_{ij} A_{ij} - \sum_{i,j} M_{ij} A_{ij} \right| \leq \sum_{i,j} A_{ij} \underbrace{|m_{ij} - M_{ij}|}_{< \frac{\varepsilon}{(b-a)(\beta-\alpha)} \text{ by } (*)} < \varepsilon,$

the inequalities $(**)$ and $(**)'$ yield

$$\left| \int_a^b \int_\alpha^\beta f(x,y)\,dy\,dx - \int_\alpha^\beta \int_a^b f(x,y)\,dx\,dy \right| < \varepsilon.$$

(This holds $\forall \varepsilon > 0$, i.e., the left-hand side $= 0$.) $\square$

Examples:

- $f(x,y) = x^y$ on $I = [0,1] \times [\alpha, \beta]$, with $0 < \alpha < \beta$. We have:

$$\int_\alpha^\beta x^y\,dy = \int_\alpha^\beta e^{y \ln x}\,dy = \frac{1}{\ln x} e^{y \ln x} \Big|_{y=\alpha}^{y=\beta} = \frac{x^\beta - x^\alpha}{\ln x}, \text{ and}$$

$\exp \ln x^y = e^{y \ln x}$

$$\int_0^1 x^y\,dx = \frac{x^{y+1}}{y+1} \Big|_{x=0}^{x=1} = \frac{1}{y+1}.$$

Thus, $\displaystyle\int_0^1 \frac{x^\beta - x^\alpha}{\ln x}\,dx = \int_\alpha^\beta \frac{1}{y+1}\,dy = \ln|y+1| \Big|_\alpha^\beta = \ln(1+\beta) - \ln(1+\alpha) = \ln \frac{1+\beta}{1+\alpha}.$

- $f(x,y) = \dfrac{x^2 - y^2}{(x^2+y^2)^2}$ on $I = [0,1] \times [\alpha, 1]$. See Problem 4, Homework 7.

Note: $f(x,y)$ is <u>not</u> continuous on $[0,1] \times [0,1]$. In fact, we show in the homework that

$$\int_0^1 \int_0^1 f(x,y)\,dx\,dy = -\frac{\pi}{4} \neq \int_0^1 \int_0^1 f(x,y)\,dy\,dx = \frac{\pi}{4}.$$