Advanced Calculus and Methods of Mathematical Physics — Session 19

Prof. Sören Petrat, Constructor University, Spring 2023 — Apr. 14, 2023

Recall: • If $\Lambda(\gamma) := \sup_{\tau} \Lambda(\tau, \gamma) < \infty$, then γ is rectifiable and $\Lambda(\gamma)$ is its length.

• If $\gamma \in C^1([a,b])$, then $\Lambda(\gamma) = \int_a^b \|\gamma'(t)\| \, dt$.

Examples:

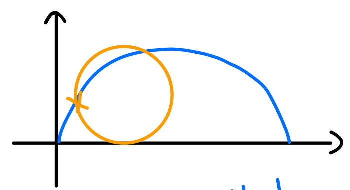
• Non-rectifiable curve: see homework.

• Circumference of a circle: $\gamma(t) = R(\cos t, \sin t)$, $t \in [0, 2\pi]$

$\Rightarrow \gamma'(t) = R(-\sin t, \cos t) \Rightarrow \|\gamma'(t)\| = \sqrt{R^2 \sin^2 t + R^2 \cos^2 t} = R.$

$\Rightarrow \Lambda(\gamma) = \int_0^{2\pi} R \, dt = 2\pi R.$

• Cycloid: $\gamma(t) = (t - \sin t, 1 - \cos t)$, $t \in [0, 2\pi]$



"move on a circle that is moved along x-axis"

$\Rightarrow \gamma'(t) = (1 - \cos t, \sin t)$

$\Rightarrow \|\gamma'(t)\|^2 = (1 - \cos t)^2 + \sin^2 t = 1 - 2\cos t + \cos^2 t + \sin^2 t = 2(1 - \cos t)$

standard trigonometric identity $= 4 \sin^2\left(\frac{t}{2}\right)$

$\Rightarrow \Lambda(\gamma) = \int_0^{2\pi} 2 \sin\left(\frac{t}{2}\right) dt = -4\cos\left(\frac{t}{2}\right)\Big|_0^{2\pi} = 4 - (-4) = 8.$

Next: Define the integral of functions along curves.

Heuristically: For $f: \mathbb{R}^n \to \mathbb{R}$: $\underbrace{\int_\gamma f \, ds \quad \text{with } \text{"} ds = \|d\gamma\| = \|\frac{d\gamma}{dt}\| dt \text{."}}_{\text{Sum up the values of } f \text{ along the curve } \gamma.}$

Definition: For $f \in C(\gamma, \mathbb{R})$, and $\gamma: [a,b] \to \mathbb{R}^n$ a C^1 curve, we define the

$\boxed{\text{line integral}}$ $\int_\gamma f \, ds := \int_a^b f(\gamma(t)) \, \|\gamma'(t)\| \, dt.$

Note: • $\int_\gamma f \, ds$ is independent of the parametrization of γ (see HW).

• For $f = 1$ we get $\int_\gamma ds = \Lambda(\gamma)$.

One useful parametrization is the arclength parametrization:

Define $s(t) = \int_a^t \|\gamma'(\tau)\| \, d\tau$. Then $s'(t) = \|\gamma'(t)\| > 0$ for $\gamma'(t) \neq 0$

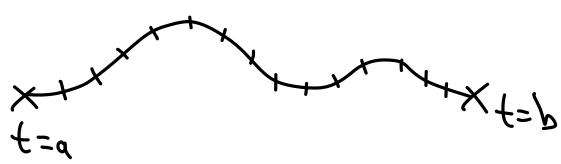
("non-degenerate parametrization").

Since $s(t)$ is monotonic it is invertible, with inverse $t(s)$.

We call $\gamma(t(s))$ the $\boxed{\text{arclength parametrization.}}$

$$\Rightarrow \frac{d}{ds} \gamma(t(s)) = \gamma'(t(s)) \frac{dt(s)}{ds} = \gamma'(t(s)) \frac{1}{s'(t(s))} \Rightarrow \left\| \frac{d}{ds} \gamma(t(s)) \right\| = \left\| \frac{\gamma'}{\|\gamma'\|} \right\| = 1,$$

i.e., we go through the curve with speed 1.



With this parametrization we find

$$\int_\gamma f \, ds := \int_a^b f(\gamma(t)) \, \|\gamma'(t)\| \, dt = \int_0^{\Lambda(\gamma)} f(\gamma(t(s))) \, ds$$

substitution $t = t(s)$

$\Rightarrow \frac{dt}{ds} = \frac{1}{s'(t)} = \frac{1}{\|\gamma'(t)\|}$

$\Rightarrow ds = \|\gamma'(t)\| dt$

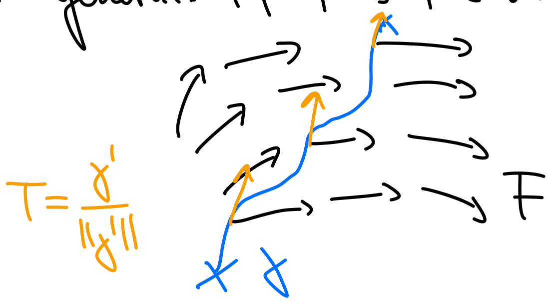
Next: How to def. line integrals for vector fields $F: D \subset \mathbb{R}^n \to \mathbb{R}^n$?
(E.g., force fields in physics.)

Motivation: In physics: work $=$ force $\cdot$ $\overbrace{\text{length}}^{\text{or rather "displacement"}}$.

In 1 dim.: $W = \int F(s)\,ds$.

In n dimensions: Only displacement in the direction of the force is work (e.g., displacement orthogonal to force causes no work).

In general: If T is the unit tangent vector, then work $= \langle F, T \rangle$.



$$\Rightarrow \text{Total work } W = \int_\gamma \langle F, T \rangle\,ds = \int_\gamma \langle F, \frac{\gamma'}{\|\gamma'\|} \rangle\,ds$$

$$= \int_a^b \langle F(\gamma(t)), \frac{\gamma'(t)}{\|\gamma'(t)\|} \rangle \, \|\gamma'(t)\|\,dt$$

$$= \int_a^b \langle F \circ \gamma, \gamma' \rangle\,dt$$

$$= \int_a^b F(\gamma(t)) \cdot \gamma'(t)\,dt.$$

<u>Definition:</u> For $F \in C(\gamma, \mathbb{R}^n)$ and γ a C^1 curve, we define the **line integral** $\int_\gamma F\,ds = \int_a^b F(\gamma(t))\,\gamma'(t)\,dt$.

Note: $F\,dx := F_1\,dx_1 + \dots + F_n\,dx_n$ is called a first-order differential form.

Example: F = Coulomb force $= \lambda \frac{\vec{x}}{\|\vec{x}\|^3}$ $\quad (F: \mathbb{R}^3 \to \mathbb{R}^3)$

For going in a circle, the work should be 0 (curve always orthogonal to F).

Indeed: $\gamma: [0, 2\pi) \to \mathbb{R}^3$, $\gamma(t) = \begin{pmatrix} \cos t \\ \sin t \\ 0 \end{pmatrix}$ (say, circle in x-y plane) $\Rightarrow \gamma'(t) = \begin{pmatrix} -\sin t \\ \cos t \\ 0 \end{pmatrix}$

$$\Rightarrow \int_\gamma F \, ds = \int_0^{2\pi} \lambda \begin{pmatrix} \cos t \\ \sin t \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -\sin t \\ \cos t \\ 0 \end{pmatrix} dt = 0$$

Work to bring a particle from $x = 1$ to $x = \infty$ on a straight line:

$\tilde{\gamma}: [1, \infty) \to \mathbb{R}^3$, $\tilde{\gamma}(t) = \begin{pmatrix} t \\ 0 \\ 0 \end{pmatrix} \Rightarrow \tilde{\gamma}'(t) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$$\Rightarrow \text{Work } W = \int_{\tilde{\gamma}} F \, ds = \int_1^\infty \lambda \frac{\begin{pmatrix} t \\ 0 \\ 0 \end{pmatrix}}{t^3} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} dt = \lambda \int_1^\infty \frac{1}{t^2} dt = \lambda.$$

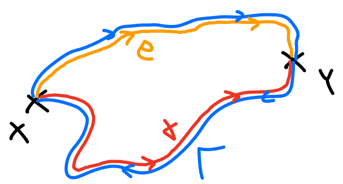
Generally, the value of $\int_\gamma F \, dx$ might depend on all of γ. But sometimes it might just depend on $\gamma(a)$ and $\gamma(b)$ (and not on $\gamma(t)$ for $a < t < b$).

Definition: Let $F \in C(D, \mathbb{R}^n)$, $D \subset \mathbb{R}^n$ a domain, γ piecewise $C^1([a,b], \mathbb{R}^n)$ ("piecewise smooth"). Then we call F **conservative** if $\int_\gamma F \, dx$ depends only on $\gamma(a)$ and $\gamma(b)$.

Note: In the differential form language: F conservative $\Leftrightarrow$ $F \, dx$ exact.

Lemma: F conservative $\Leftrightarrow \int_\gamma F \, dx = 0 \ \forall$ closed γ.

Proof:



$$\int_\Gamma F \, dx = \int_e F \, dx - \int_\gamma F \, dx$$

If $\int_\gamma F \, dx$ depends only on $\gamma(a) = x$ and $\gamma(b) = y$, then $\int_e F \, dx = \int_\gamma F \, dx$, since $\gamma(a) = e(a)$, $\gamma(b) = e(b)$. Thus $\int_\Gamma F \, dx = 0$.

If $\int_\Gamma F \, dx = 0$, then $\int_e F \, dx = \int_\gamma F \, dx \ \forall e$, so $\int_\gamma F \, dx$ depends only on $\gamma(a), \gamma(b)$. $\square$

Note: This is extremely useful for computations. Suppose F is conservative, and I need to compute $\int_\gamma F ds$ for a complicated path γ. Then

$\int_\gamma F ds = \int_{\tilde{\gamma}} F ds$ for any $\tilde{\gamma}$ with the same start/end points. This $\tilde{\gamma}$ can be chosen to be much more convenient for our computation.

(We will come back to this once we have good criteria to find out whether F is conservative.)