Constructor University — March 30, 2023
Spring 2023

Topology and Manifolds

Homework 8

Due on April 13, 2023, before class

Problem 1 [**4 points**]

Let M be the open submanifold of $\mathbb{R}^2$ where both x and y are positive, and let $F : M \to M$ be the map $F(x, y) = (xy, \frac{y}{x})$. Show that F is a diffeomorphism, and compute F_*X and F_*Y, where

$$X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \quad Y = y\frac{\partial}{\partial x}.$$

Problem 2 [**3 points**]

Compute the flow of the vector field $X = x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y}$ on $\mathbb{R}^2$.

Problem 3 [**4 points**]

For any integer $n \geq 1$, by identifying $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ in the usual way, we can consider the odd-dimensional sphere $\mathbb{S}^{2n-1}$ as a subset of $\mathbb{C}^n$. Define a global flow on $\mathbb{S}^{2n-1}$ by $\theta(t, z) = e^{it}z$. Show that the infinitesimal generator of θ is a smooth nowhere-vanishing vector field on $\mathbb{S}^{2n-1}$. For $n = 2$, find the integral curves of θ.

Problem 4 [**5 points**]

Let X and Y be two (smooth) vector fields on a smooth manifold M. Let θ_t be the local flow of Y. Prove that

$$\left.\frac{d}{dt}\right|_{t=0} \big((\theta_{-t})_* X\big) = [Y, X],$$

where $(\theta_{-t})_*$ is the pushforward of $\theta_{-t} : M \to M$, and $[X, Y]$ the Lie bracket. Start with the fact that for any smooth $f : M \to \mathbb{R}$ one can write

$$f(\theta_t(x)) = f(x) + t(Yf)(x) + t^2 E(x, t)$$

for some smooth $E(x, t)$ with $E(x, 0) = \frac{1}{2}(Y^2 f)(x)$, and then compute directly.

Problem 5 [**4 points**]

Consider the smooth manifold $M = \{(x, y) \in \mathbb{R}^2 : x > 0\}$, and the smooth function $f : M \to \mathbb{R}$, $f(x, y) = \frac{x}{x^2+y^2}$. Compute the coordinate representation for df and determine the set of all points $p \in M$ at which $df_p = 0$, once in standard coordinates (x, y), and once in polar coordinates (r, φ).