Constructor University
Spring 2023

April 13, 2023

Topology and Manifolds

Homework 9

Due on April 20, 2023, before class

Problem 1 [3 points]

Let M be a smooth manifold, and $\omega \in T_p^*M$. Suppose two local coordinates (x^j) and $(\tilde{x}^j)$ are given for some neighborhood of $p \in M$. How do the components of ω in one set of coordinates change when expressed in the other coordinates?

Problem 2 [3 points]

Let V be a vector space, then $V^* \otimes \ldots \otimes V^*$ (k times) is called the space of covariant k-tensors. A tensor T is called alternating if $T(\ldots, v_i, \ldots, v_j, \ldots) = -T(\ldots, v_j, \ldots, v_i, \ldots)$, and symmetric if $T(\ldots, v_i, \ldots, v_j, \ldots) = T(\ldots, v_j, \ldots, v_i, \ldots)$.

Now consider $V = \mathbb{R}^3$. Let (e^1, e^2, e^3) (note the upper indices) be the standard dual basis for $(\mathbb{R}^3)^*$.

(a) Show that $e^1 \otimes e^2 \otimes e^3$ is not equal to a sum of an alternating tensor and a symmetric tensor.

(b) Can $e^1 \otimes e^2 + e^2 \otimes e^1 \in V^* \otimes V^*$ be written as $w^1 \otimes w^2$ for some $w^1, w^2 \in V^*$? (Prove your answer.)

Problem 3 [3 points]

(a) Let $\omega \in \Lambda^k(V^*)$, $\eta \in \Lambda^\ell(V^*)$. Prove that $\omega \wedge \eta = (-1)^{k\ell}\, \eta \wedge \omega$.

(b) Let $\omega^1, \ldots, \omega^k \in \Lambda^1(V^*)$ and $v_1, \ldots, v_k \in V$. Prove that

$$\omega^1 \wedge \ldots \wedge \omega^k(v_1, \ldots, v_k) = \det \omega^j(v_i).$$

Problem 4 [5 points]

Let $F : M \to N$ be a smooth map between smooth manifolds M, N, and let ω, η be differential forms on N, then the pullbacks $F^*\omega$ and $F^*\eta$ are differential forms on M.

(a) Prove that $F^*(\omega \wedge \eta) = (F^*\omega) \wedge (F^*\eta)$.

(b) Prove that in any smooth chart,

$$F^*\left(\sum_I{}' \omega_I\, dy^{i_1} \wedge \ldots \wedge dy^{i_k}\right) = \sum_I{}' (\omega_I \circ F)\, d(y^{i_1} \circ F) \wedge \ldots \wedge d(y^{i_k} \circ F).$$

(c) Let $(U, (x^i))$ and $(\tilde{U}, (\tilde{x}^j))$ be overlapping smooth coordinate charts on the smooth n-manifold M. Prove that on $U \cap \tilde{U}$ we have

$$d\tilde{x}^1 \wedge \ldots \wedge d\tilde{x}^n = \det\left(\frac{\partial \tilde{x}^j}{\partial x^i}\right) dx^1 \wedge \ldots \wedge dx^n.$$

Problem 5 [6 points]

We consider the manifold $\mathbb{R}^n$. Recall that for a k-form ω on $\mathbb{R}^n$ we define the exterior derivative $d\omega$ as the $(k+1)$-form

$$d\left({\sum_J}' \omega_J dx^J\right) = {\sum_J}' d\omega_J \wedge dx^J,$$

where $d\omega_J$ is the differential of $\omega_J : \mathbb{R}^n \to \mathbb{R}$. Prove that d has the following properties:

(a) d is $\mathbb{R}$-linear.

(b) For a smooth k-form ω and a smooth ℓ-form η we have

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta.$$

(c) $d \circ d = 0$.

(d) Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a smooth map and ω a smooth k-form on $\mathbb{R}^m$, then

$$F^*(d\omega) = d(F^*\omega).$$