Constructor University April 20, 2023
Spring 2023

Topology and Manifolds

Homework 10

Due on April 27, 2023, before class

Problem 1 [4 points]
On $\mathbb{R}^3$, consider the 2-form

$$\Omega = xdy \wedge dz + ydz \wedge dx + zdx \wedge dy.$$

(a) Compute Ω in spherical coordinates (ρ, φ, θ) defined by

$$(x, y, z) = (\rho \sin\varphi \cos\theta, \rho \sin\varphi \sin\theta, \rho\cos\theta).$$

(b) Compute $\mathrm{d}\Omega$ in both Cartesian and spherical coordinates and verify that both expressions represent the same 3-form.

Problem 2 [2 points]
Let $F : \mathbb{R}^2 \to \mathbb{R}^3, F(u, v) = (u^2, v^3, e^u - v)$ and let ω be the 2-form $\omega = xdx \wedge dy + zdx \wedge dz$ on $\mathbb{R}^3$. Compute the pullback $F^*\omega$.

Problem 3 [8 points]
Suppose M, N are nonempty oriented smooth n-manifolds, and ω, η are compactly supported n-forms on M. Prove the following properties:

(a) Linearity: If $a, b \in \mathbb{R}$, then

$$\int_M (a\omega + b\eta) = a\int_M \omega + b\int_M \eta.$$

(b) Orientation reversal: If $-M$ denotes M with the opposite orientation, then $\int_{-M} \omega = -\int_M \omega$.

(c) Positivity: If ω is nonvanishing and positively oriented, then $\int_M \omega > 0$.

(d) Diffeomorphism invariance: If $F : N \to M$ is an orientation-preserving diffeomorphism, then

$$\int_M \omega = \int_N F^*\omega.$$

Problem 4 [6 points]

Let G be a Lie group acting on a smooth manifold M, and let α be a differential 1-form on M. The group action via g is denoted by $T_g : M \to M$, i.e., $T_g(m) = g \cdot m$. We say that α is G-invariant when $T_g^*\alpha = \alpha$ for all $g \in G$.

(a) Show that the differential 1-forms

$$\alpha = x\,\mathrm{d}x + y\,\mathrm{d}y, \qquad \text{and} \quad \beta = x\,\mathrm{d}y - y\,\mathrm{d}x$$

on $\mathbb{R}^2$ are invariant under the group of rotations $SO(2)$.

(b) Let ω be a differential form on $\mathbb{R}^2 \setminus \{0\}$ which is invariant under $SO(2)$. Show that ω can be expressed as

$$f(r)\alpha + g(r)\beta,$$

where $r = \sqrt{x^2 + y^2}$ and f and g are smooth functions defined on $\mathbb{R}^2 \setminus \{0\}$.