Topology and Manifolds, Constructor University Bremen, Spring 2023 — Session 14, Mar. 17, 2023

Prof. Dr. Soeren Petrat

---

4. Lie Groups

Idea: connect groups and smooth manifolds

Motivation: continuous symmetries of differential eq.s, e.g., Galilean symm. in classical mechanics (translation, rotation, uniform motion), or Poincaré symm. in relativity

Recall:

Def.: A $\boxed{\text{group}}$ is a set G with an operation $\cdot : G\times G \to G$, $(x,y) \mapsto xy$ (sometimes denoted $x\cdot y$) that satisfies:

- $(xy)z = x(yz)$ (associativity)
- $\exists$ identity e, i.e. $ex = xe = x$ $\forall x \in G$ (note: unique)
- $\exists$ inverse, i.e., $\forall x \in G$ $\exists x^{-1} \in G$ with $x^{-1}x = xx^{-1} = e$ (note: unique)

If also $xy = yx$ $\forall x,y \in G$, then G is called $\boxed{\text{abelian group}}$.

If $H \subset G$ with operation $\cdot$ is also a group, $(H,\cdot)$ is called a $\boxed{\text{subgroup}}$ of G.

Def.: A $\boxed{\text{Lie group}}$ is a smooth manifold G that is also a group, with the property that

- multiplication map $\cdot : G\times G \to G$, $(x,y) \mapsto xy$, and
- inverse map $G \to G$, $x \mapsto x^{-1}$

are smooth.

Def.: Let $g \in G$, G Lie group, then the left and right translations $L_g, R_g : G \to G$ are defined as $L_g(h) = gh$, $R_g(h) = hg$.

Proposition: L_g and R_g are diffeomorphisms.

Proof: smooth as composition of smooth maps, $L_{g^{-1}}, R_{g^{-1}}$ smooth inverses. $\square$

Examples:

- additive group $(\mathbb{R}^n, +)$
  - ↳ map $(x,y) \mapsto x+y$ is smooth
  - ↳ map $x \mapsto -x$ is smooth

  } both are linear

  $\Rightarrow$ an abelian connected Lie group of dimension n

- multiplicative group $(\mathbb{R}^*, \cdot)$, where $\mathbb{R}^* = \{x \in \mathbb{R}, x \neq 0\}$
  - ↳ maps $(x,y) \mapsto x \cdot y$ and $x \mapsto \frac{1}{x}$ are smooth $(x \neq 0)$

  note: – $\mathbb{R}^*$ is not connected: $\mathbb{R}^* = \underbrace{\mathbb{R}^{>0}}_{\mathbb{R}^+} \cup \underbrace{\mathbb{R}^{<0}}_{\mathbb{R}^-}$ union of disjoint open sets

  – $\mathbb{R}^{>0}$ is a subgroup of $\mathbb{R}^*$ and open, thus itself a Lie group

- general linear group $GL_n(\mathbb{R})$ = set of invertible $n \times n$ matrices with matrix multiplication

  note: – $M_{n \times n}(\mathbb{R})$ (real $n \times n$ matrices) is a vector space, thus a smooth manifold

  – $A \in GL_n(\mathbb{R}) \Leftrightarrow \det A \neq 0$

  $\Rightarrow$ since $\det: M_{n \times n}(\mathbb{R}) \to \mathbb{R}$ is continuous, $GL_n(\mathbb{R})$ is open, thus also a smooth manifold

  – $(A,B) \mapsto A \cdot B$ smooth (matrix entries are polynomials)

  – $A \mapsto A^{-1} = \frac{1}{\det A} \underbrace{\operatorname{adj} A}_{\text{adjugate of } A, \text{ some polynomial}}$ smooth

  $\Rightarrow GL_n(\mathbb{R})$ is a Lie group

- $GL_n^+(\mathbb{R}) = \{A \in GL_n(\mathbb{R}) : \det A > 0\}$
  ↳ $\det AB = \det A \cdot \det B$ and $\det A^{-1} = \frac{1}{\det A}$ $\Rightarrow$ $GL_n^+(\mathbb{R})$ subgroup and open ($\det^{-1}((0,\infty))$)
  $\Rightarrow$ Lie group
- similar: $GL(V) := \{$invertible linear maps $V \to V\}$ for any finite dimensional vector space V is a Lie group
- circle $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ with complex multiplication is a Lie group (circle group)
  $\Rightarrow$ n-torus $S^1 \times \dots \times S^1$ is an n-dim. Lie group
- Heisenberg group $H_3 = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}$ plays important role in physics.
  One can check that it is a Lie group

Def.: Let H, G be Lie groups. A smooth map $F: G \to H$ is called **Lie group homomorphism** if
$$F(\underbrace{xy}_{\text{mult. in } G}) = \underbrace{F(x)F(y)}_{\text{mult. in } H} \quad \forall x, y \in G.$$
If F also a diffeomorphism, we call it **Lie group isomorphism**.

Ex.: • $\exp: \mathbb{R} \to \mathbb{R}^*$, $x \mapsto e^x$ Lie group homomorphism ($e^{(t+s)} = e^t e^s$)
- $\exp: \mathbb{R} \to \mathbb{R}^+$ Lie group isomorphism
- $\det: GL_n(\mathbb{R}) \to \mathbb{R}^*$ Lie group homomorphism ($\det AB = \det A \det B$)
- Lie group G, $g \in G$, then conjugation by g is the map $C_g: G \to G$, $h \mapsto ghg^{-1}$
  ↳ smooth, group homomorphism and isomorphism ($C_g^{-1} = C_{g^{-1}}$)
  $\left(C_g(h\tilde{h}) = gh\tilde{h}g^{-1} = ghg^{-1}g\tilde{h}g^{-1} = C_g(h)C_g(\tilde{h}) \right)$
  ↳ **Def.:** a subgroup $H \subset G$ is called **normal** if $C_g(H) = H$ $\forall g \in G$.

Thm.: Every Lie group homomorphism has constant rank.

Proof: Take some $g_0 \in G$ and show that dF_{g_0} has same rank as dF_e, with e=identity, by using the chain rule on $F(g_0 e) = F(L_{g_0} e)$. Details: HW. $\square$

Then the global version of the rank thm. implies:
A Lie group homomorphism is a Lie group isomorphism if and only if it is bijective.

Also note:

Proposition: Let G° be the identity component of G (the connected component containing the identity). Then G° is a Lie group, $\dim G^\circ = \dim G$, and G° is a normal subgroup of G.

Proof: HW

In some later class:

- $O(n)$, $SO(n)$, $U(n)$, $SU(n)$ as Lie groups
- representation theory: representation = Lie group homomorphism $G \to GL(V)$, V a vector space
  - e.g., $\psi(x)$ solution to a translation or rotation invariant eq., $\psi \in$ some vector space (e.g., function spaces, Hilbert spaces)

    $\Rightarrow \psi(Rx)$, $R \in G$, is a different solution

    $\psi(Rx) = \underbrace{\Pi(R)}_{\text{representation}} \psi(x)$ $\Rightarrow$ important research topic