Topology and Manifolds, Constructor University Bremen, Spring 2023 — Session 23, Apr. 27, 2023

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Recall: M a smooth oriented n-manifold, ω an n-form with compact support

If $\operatorname{supp}\omega$ covered by one chart (U,φ) then we def. $\int_M \omega := \int_{\varphi(U)} (\varphi^{-1})^* \omega$.

This def. is independent of the choice of chart.

More generally, if

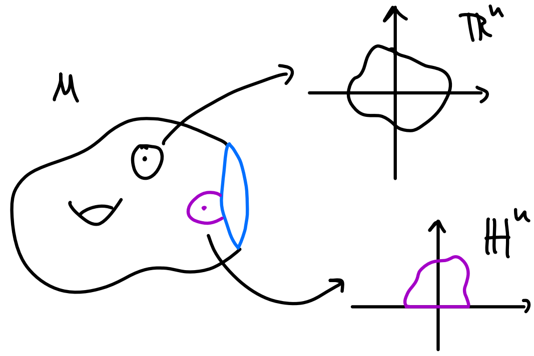
- $\{U_i\}$ finite open cover of $\operatorname{supp}\omega$ with positively oriented charts (U_i, φ_i)
- $\{\psi_i\}$ a smooth partition of unity subordinate to $\{U_i\}$

Def.: $\int_M \omega := \sum_i \underbrace{\int_M \psi_i \omega}_{\text{well-def. on single chart } (U_i,\varphi_i)}$

Proposition: This def. is independent of the choice of open cover or partition of unity.

Next: Stokes thm. $\Rightarrow$ we need manifold boundaries first

5.8 Manifolds with Boundary



Def.: $\mathbb{H}^n = \{(x^1, \dots, x^n) \in \mathbb{R}^n : x^n \geq 0\}$ (upper half space)

$\partial \mathbb{H}^n = \{(x^1, \dots, x^{n-1}, 0) \in \mathbb{R}^n\}$

An $\boxed{\text{n-manifold with boundary}}$ is a second-countable Hausdorff space M where every $p \in M$ has a neighborhood homeomorphic to either an open subset of $\mathbb{R}^n$ or $\mathbb{H}^n$.

- $p \in M$ is a boundary point if it is in the domain of a boundary chart ($\varphi(U) \cap \partial \mathbb{H}^n \neq \emptyset$) that maps p to $\partial \mathbb{H}^n$
- ∂M = set of all boundary points

Note:
- Any $p \in M$ is either a boundary point or an interior point ($\varphi(U)$ open subset of $\mathbb{R}^n$)
  $\Rightarrow$ ∂M and $\mathrm{Int}\, M$ are disjoint
- ∂M refers to manifold boundary, i.e., all $p \in M$ covered by a boundary chart
  $\to$ this is not necessarily equal to topological boundary (if we think of $M \subset$ some other topological space)
- ∂M = $(n-1)$-manifold without boundary (e.g., $\partial \overline{\mathbb{B}}^n = S^{n-1}$)
- Many results we discussed for manifolds without boundary also hold for manifolds with boundary (see Lee's book for details)
- Smooth structures are def. analogously as for manifolds without boundary
  (transition map $U \to \mathbb{R}^n$, $U \subset \mathbb{H}^n$ open, is smooth if it can be extended to a smooth map $\hat{U} \to \mathbb{R}^n$ with $\hat{U} \subset \mathbb{R}^n$ open)

$\Rightarrow$ Def. of $\boxed{\text{smooth manifolds with boundary}}$

<u>Proposition:</u> If manifold with boundary M is orientable, then also ∂M is orientable.

Note: • ∂M is actually an embedded submanifold in M with codim. 1 (= embedded hypersurface).
One can show that there is always a smooth outward-pointing vector field along ∂M.

In fact: If manifold with boundary M is orientable, then all outward-pointing vector fields along ∂M determine the same orientation on ∂M, which is called induced orientation or Stokes orientation

Example:

- S^n is orientable as the boundary of $\overline{\mathbb{B}}_n$ (closed unit ball)
- What orientation does $\partial \mathbb{H}^n$ inherit from $\mathbb{H}^n$ ($\mathbb{H}^n$ has standard orientation inherited from $\mathbb{R}^n$)?
  - $(x^1, \dots, x^{n-1}, 0) \in \partial \mathbb{H}^n$ identified with $(x^1, \dots, x^{n-1}) \in \mathbb{R}^{n-1}$
  - $-\frac{\partial}{\partial x^n}$ is an outward pointing vector field along $\partial \mathbb{H}^n$

$$\Rightarrow \left[-\frac{\partial}{\partial x^n}, \frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^{n-1}}\right] = -\left[\frac{\partial}{\partial x^n}, \frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^{n-1}}\right] = (-1)^n \left[\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\right]$$

$$\Rightarrow \text{Induced orientation on } \partial \mathbb{H}^n = \begin{cases} \text{standard orientation on } \mathbb{R}^{n-1} & \text{if } n \text{ even} \\ \text{opposite of standard orientation on } \mathbb{R}^{n-1} & \text{if } n \text{ odd} \end{cases}$$

5.9 Stokes Theorem

<u>Thm.:</u> Let M be an oriented smooth n-manifold with boundary and ω a compactly supported smooth $(n-1)$-form on M. Then

$$\boxed{\int_M d\omega = \int_{\partial M} \omega \qquad \text{(Stokes Theorem)}}$$

Remarks:
- $d\omega$ = exterior derivative of ω = n-form
- ∂M has orientation induced by M (Stokes orientation)
- ω on right-hand side means $i_{\partial M}^{*}\omega$ ($i_{\partial M}: \partial M \to M$ inclusion)