Topology and Manifolds, Constructor University Bremen, Spring 2023 Session 26
Prof. Dr. Soeren Petrat May 5, 2023

An example of an infinite dim. symplectic vector space.

Let $\mathcal{H}$ be a Hilbert space, let us take $\mathcal{H} = L^2(\mathbb{R}^d)$ to be specific.

$\Rightarrow$ scalar product $\langle f,g\rangle = \int \overline{f(x)}\, g(x)\,dx$.

Now let $\overline{f}$ be def. by $\overline{f}(x) = \overline{f(x)}$ (complex conjugation), and def. the vector space $Y = \{(f,\overline{f}),\ f\in\mathcal{H}\}$.

We def. $\sigma((f,\overline{f}),(g,\overline{g})) = \operatorname{Im}\langle f,g\rangle$

$\hookrightarrow$ nondegenerate

$\hookrightarrow \sigma((g,\overline{g}),(f,\overline{f})) = \operatorname{Im}\langle g,f\rangle = \operatorname{Im}\overline{\langle f,g\rangle} = -\operatorname{Im}\langle f,g\rangle = -\sigma((f,\overline{f}),(g,\overline{g}))$

$\Rightarrow (Y,\sigma)$ is a symplectic vector space

Note:

$$\left\langle (f,\overline{f}), \frac{1}{2i}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}g\\\overline{g}\end{pmatrix}\right\rangle = \frac{1}{2i}\left(\langle f,g\rangle - \langle \overline{f},\overline{g}\rangle\right) = \frac{1}{2i}\left(\langle f,g\rangle - \langle g,f\rangle\right) = \operatorname{Im}\langle f,g\rangle$$

$$= \sigma((f,\overline{f}),(g,\overline{g})).$$

If $R: Y\to Y$ is symplectic, then $\sigma(R(f,\overline{f}),R(g,\overline{g})) = \sigma((f,\overline{f}),(g,\overline{g}))$.

$$\Rightarrow R^*\begin{pmatrix}1&0\\0&-1\end{pmatrix}R = \begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

Let us consider specifically $R = \begin{pmatrix}u&\overline{v}\\v&\overline{u}\end{pmatrix}$, with $u,v:\mathcal{H}\to\mathcal{H}$ (bounded), $\overline{u}f := \overline{u\overline{f}}$.

Then $R^*\begin{pmatrix}1&0\\0&-1\end{pmatrix}R = \begin{pmatrix}1&0\\0&-1\end{pmatrix} \iff \begin{pmatrix}u^*&v^*\\\overline{v}^*&\overline{u}^*\end{pmatrix}\underbrace{\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}u&\overline{v}\\v&\overline{u}\end{pmatrix}}_{=\begin{pmatrix}u&\overline{v}\\-v&-\overline{u}\end{pmatrix}} = \begin{pmatrix}1&0\\0&-1\end{pmatrix}$

$\Rightarrow$ R is symplectic $\Longleftrightarrow$ $\left.\begin{array}{l} u^* u - v^* v = 1 \;,\; \bar{u}^* \bar{u} - \bar{v}^* \bar{v} = 1 \,, \\ u^* \bar{v} - v^* \bar{u} = 0 \;,\; \bar{v}^* u - \bar{u}^* v = 0 \end{array}\right\} (*)$

An R that satisfies (*) is called Bogoliubov map.

A few properties of Bogoliubov maps:

- $u^* u = 1 + v^* v \geq 1 \Rightarrow u$ is invertible
- Specific example for $L^2([0,1])$: $u = \sum_{k=-\infty}^{\infty} \sigma_k |e^{ikx}\rangle\langle e^{ikx}|$, $\sigma_k \in \mathbb{R}$

$$v = \sum_{k=-\infty}^{\infty} (-\gamma_k) |e^{ikx}\rangle\langle e^{ikx}| \quad , \gamma_k \in \mathbb{R}$$

with $\sigma_k^2 - \gamma_k^2 = 1$, $|\sigma_k| \geq 1$, i.e., $\sigma_k = \cosh \lambda_k$, $\gamma_k = \sinh \lambda_k$ for some $\lambda_k \in \mathbb{R}$.

Recall: A linear map $f: V \to V$ is symplectic if and only if $A^T J A = J$, where A is the matrix of f and J is the matrix of the symplectic form ω of V.

Now: Suppose f (with matrix A) and g (with matrix B) are symplectic.

Then $(AB)^T J (AB) = B^T \underbrace{A^T J A}_{=J} B = B^T J B = J$, so also AB is symplectic.

$\Rightarrow$ Symplectic maps form a group

Recall:

<u>Def.</u>: • A 2-form ω on a smooth manifold M is called nondegenerate if ω_p is a nondegenerate 2-covector for each $p \in M$.

• A **symplectic form** on M is a $\overbrace{\text{closed}}^{\text{exterior derivative}=0}$ nondegenerate 2-form.

• M with a $\underbrace{\text{symplectic form}}_{\text{or symplectic structure}}$ is called **symplectic manifold.**

Note: • From the discussion above we know that $\dim M$ must be even.

• Also: If ω is a symplectic form on a $2n$-dim. manifold M, then ω^n is a nonvanishing $2n$-form, so every symplectic manifold is orientable.

• If (M_1, ω_1) and (M_2, ω_2) are symplectic manifolds, then a diffeomorphism $F: M_1 \to M_2$ with $F^* \omega_2 = \omega_1$ is called symplectomorphism $\Rightarrow$ symplectic geometry / topology

Ex.: • $\mathbb{R}^{2n}$ with coordinates $(x^1, \dots, x^n, y^1, \dots, y^n)$:

$\omega = \sum_{i=1}^{n} dx^i \wedge dy^i$ is symplectic (closed: ✓ nondegenerate: ✓)

• Σ an oriented smooth 2-manifold, ω nonvanishing 2-form.

$\Rightarrow \omega$ closed ($d\omega = 3$-form), $\omega = \alpha^1 \wedge \beta^1$ locally, i.e., nondegenerate

Globally, we have the following very strong result.

<u>Theorem (Darboux)</u>: Let (M, ω) be a $2n$-dim. symplectic manifold. Then for any $p \in M$ there are smooth coordinates $(x^1, \dots, x^n, y^1, \dots, y^n)$ centered at p in which ω has coordinate representation

$$\omega = \sum_{i=1}^{n} dx^i \wedge dy^i .$$

Proof: Uses time-dependent flows, we skip it here.

Note: Such coordinates are called Darboux/symplectic/canonical coordinates.

Next:

<u>Def.:</u> Given a symplectic manifold (M,ω) and a smooth function $f \in C^\infty(M)$, we def. the Hamiltonian vector field of f to be X_f, def. via

$$X_f \lrcorner\, \omega = df \quad \text{, or ,} \quad \omega(X_f, Y) = df(Y) = Yf \text{ for any vector field } Y.$$

In Darboux coordinates:

$$X_f = \sum_{i=1}^n \left(a^i \frac{\partial}{\partial x^i} + b^i \frac{\partial}{\partial y^i} \right) \quad , a^i, b^i \text{ some coefficient functions}$$

$$\hookrightarrow X_f \lrcorner\, \omega = \sum_{i=1}^n \left(a^i \frac{\partial}{\partial x^i} + b^i \frac{\partial}{\partial y^i} \right) \lrcorner \sum_{j=1}^n dx^i \wedge dy^i$$

$$= \sum_{i=1}^n \left(a^i dy^i - b^i dx^i \right)$$

$$\hookrightarrow df = \sum_{i=1}^n \left(\frac{\partial f}{\partial x^i} dx^i + \frac{\partial f}{\partial y^i} dy^i \right)$$

$$\Longrightarrow a^i = \frac{\partial f}{\partial y^i} \quad , b^i = -\frac{\partial f}{\partial x^i}$$

$$\Longrightarrow X_f = \sum_{i=1}^n \left(\frac{\partial f}{\partial y^i} \frac{\partial}{\partial x^i} - \frac{\partial f}{\partial x^i} \frac{\partial}{\partial y^i} \right)$$

Note: $X_f f = df(X_f) = \omega(X_f, X_f) = 0$, so f does not change in the direction X_f

$\Longrightarrow f$ is constant along integral curves of X_f.

<u>Def.:</u> A symplectic manifold (M,ω) together with $H \in C^\infty(M)$ is called Hamiltonian system, and
- H is called Hamiltonian
- the flow of X_H is called Hamiltonian flow
- the integral curves of X_H are called trajectories or orbits

Trajectories $\gamma(t)=(x^i(t),y^i(t))$ satisfy $\cdot\ \dot{x}^i(t)=\frac{\partial H}{\partial y^i}(x(t),y(t))$
$\cdot\ \dot{y}^i(t)=-\frac{\partial H}{\partial x^i}(x(t),y(t))$ } Hamilton's eq.s

derivative with respect to t

Example: Classical mechanics, n-body problem

- n particles in $\mathbb{R}^3$: $(q_1(t),\dots,q_n(t))\in\mathbb{R}^{3n}$, $q_i(t)\in\mathbb{R}^3$
- configuration space $Q=\mathbb{R}^{3n}\setminus C$, $C=\{q\in\mathbb{R}^{3N},\ q_k=q_\ell \text{ for some } k\neq\ell\}$
  $=$ collision set
- Newton's eq.s: $m\ddot{q}_k(t)=F_k(q(t))$ for some smooth forces $F\in\mathbb{R}^{3n}$.
- Assume F conservative, take F_k as components of covector field on Q: $F=-dV$, for some $V\in C^\infty(Q)$
- Consider cotangent space T^*Q with coordinates (q^i,p_i), then $q(t)$ satisfies Newton's law if $\gamma(t)=(q(t),p(t))$ in T^*Q satisfies

$$\dot{q}^i(t)=\frac{1}{m}\mathbb{1}^{ij}p_j(t)$$
$$\dot{p}_i(t)=-\frac{\partial V}{\partial q^i}(q(t))$$

$\Rightarrow$ Hamiltonian $H(q,p)=\frac{1}{2m}p^2+V(q)$

Note: $\cdot$ $H=$ const along flow $\Rightarrow$ conservation of energy.
- Further constraints may lead to considering different configuration spaces Q.