Constructor University
Fall 2024

December 17, 2024

Elements of Linear Algebra
Final Exam

Instructions:

- The exam has 16 multiple choice questions (several answers can be correct!) and 3 longer questions. The total number of points is 122.
- For the multiple choice questions, it is sufficient to mark the final answer(s) only. (No solution steps necessary.) There are no negative points, but of course there are fewer points if wrong answers are selected, or if right answers are not selected.
- For the longer exercises 17, 18, and 19, you need to show your work, i.e., carefully write down the steps of your solution. You will receive points not just based on your final answer, but on the correct steps in your solution.
- No tools or other resources are allowed for this exam. In particular, no notes and no calculators.
- You are free to refer to any results proven in class or the homework sheets unless stated otherwise (and unless the problem is to reproduce a result from class or the homework sheets).

Code of Academic Integrity

I pledge that the answers of this exam are my own work without the assistance of others or the usage of unauthorized material or information.

Sign to confirm that you adhere to the Academic Integrity Code:

Name: ____

Signature: ____________________________________

Matric./Student No.: __________________________

1. **(6 points)** Consider the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Which of the following is true?

A. The eigenvalue $\lambda = 1$ has algebraic multiplicty 1.
B. * The eigenvalue $\lambda = 1$ has algebraic multiplicty 2.
C. * The eigenvalue $\lambda = 1$ has geometric multiplicty 1.
D. The eigenvalue $\lambda = 1$ has geometric multiplicty 2.
E. The matrix is diagonalizable.
F. * The matrix is NOT diagonalizable.

2. **(4 points)** Consider the set

$$P = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : 7x + 2y - 5z = 0 \right\} \subset \mathbb{R}^3.$$

Note that P describes a plane through the origin, and is thus a vector space itself. Which of the following is a basis for P? Note that only one answer is correct here.

A. $\begin{pmatrix} 5 \\ 0 \\ -7 \end{pmatrix}, \begin{pmatrix} 0 \\ 5 \\ 2 \end{pmatrix}.$
B. $\begin{pmatrix} 5 \\ 0 \\ -7 \end{pmatrix}, \begin{pmatrix} 0 \\ -5 \\ 2 \end{pmatrix}.$
C. $\begin{pmatrix} 5 \\ 0 \\ 7 \end{pmatrix}, \begin{pmatrix} 10 \\ 5 \\ 2 \end{pmatrix}.$
D. $\begin{pmatrix} 5 \\ 0 \\ 7 \end{pmatrix}, \begin{pmatrix} 10 \\ 0 \\ 14 \end{pmatrix}.$
E. $\begin{pmatrix} 5 \\ 1 \\ 7 \end{pmatrix}, \begin{pmatrix} 10 \\ 6 \\ 14 \end{pmatrix}.$
F. * $\begin{pmatrix} 5 \\ 0 \\ 7 \end{pmatrix}, \begin{pmatrix} 0 \\ 5 \\ 2 \end{pmatrix}.$

3. **(4 points)** Compute the determinant of the matrix

$$A = \begin{pmatrix} 3 & 0 & 2 \\ 1 & 2 & 4 \\ 2 & 1 & 1 \end{pmatrix}.$$

A. $\det A = 0$.
B. $\det A = 2$.
C. $\det A = 14$.
D. * $\det A = -12$.
E. $\det A = 8$.
F. $\det A = -6$.

4. **(6 points)** Consider the matrix

$$A = \begin{pmatrix} 2 & 2 & 2 & 2 \\ 17/10 & 1/10 & -17/10 & -1/10 \\ 3/5 & 9/5 & -3/5 & -9/5 \end{pmatrix}.$$

A has a singular value decomposition $A = U\Sigma V^*$ with

$$U = \frac{1}{5}\begin{pmatrix} 5 & 0 & 0 \\ 0 & 3 & -4 \\ 0 & 4 & 3 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{pmatrix}, \quad V^* = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix}.$$

Which of the following statements are true?

A. $\text{rank}(A) = 3$, $\text{nullity}(A) = 0$.
B. $\text{rank}(A) = 3$, $\text{nullity}(A) = 4$.
C. * $\text{rank}(A) = 3$, $\text{nullity}(A) = 1$.
D. $\text{rank}(A) = 4$, $\text{nullity}(A) = 3$.
E. V is Hermitian.
F. * V is unitary.

5. **(6 points)** Consider the system of linear equations

$$\begin{aligned} x_1 + 2x_2 &= 1, \\ 3x_1 + \alpha x_2 &= 3, \end{aligned}$$

with parameter $\alpha \in \mathbb{R}$. Which of the following statements are true?

A. For $\alpha = 6$, the system of equations has no solutions.

B. For $\alpha = 6$, the system of equations has a unique solution.

C. * For $\alpha = 6$, the system of equations has infinitely many solutions.

D. For $\alpha \neq 6$, the system of equations has no solutions.

E. * For $\alpha \neq 6$, the system of equations has a unique solution.

F. For $\alpha \neq 6$, the system of equations has infinitely many solutions.

6. **(6 points)** A system of linear equations $Ax = b$ has been brought, through Gaussian elimination, into the reduced row-echelon form (in augmented matrix notation)

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 4 & 5 \\ 0 & 1 & 0 & -2 & 7 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

Which of the following statements are true?

A. The general solution to this system can be written as $x = \begin{pmatrix} 4 \\ -2 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ 7 \\ 3 \\ 0 \end{pmatrix}$, for $\lambda \in \mathbb{R}$.

B. * The general solution to this system can be written as $x = \begin{pmatrix} 5 \\ 7 \\ 3 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -2 \\ 2 \\ -1 \end{pmatrix}$, for $\lambda \in \mathbb{R}$.

C. The general solution to this system can be written as $x = \begin{pmatrix} 5 \\ 7 \\ 3 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -2 \\ 2 \\ 0 \end{pmatrix}$, for $\lambda \in \mathbb{R}$.

D. * The kernel of A is span $\left\{ \begin{pmatrix} 4 \\ -2 \\ 2 \\ -1 \end{pmatrix} \right\}$.

E. The kernel of A is 3.

F. * The rank of A is 3.

7. **(6 points)** Consider the vectors

$$a = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}, \quad \text{and} \quad b = \begin{pmatrix} 3 \\ -2 \\ -1 \end{pmatrix}.$$

Which of the following statements are true?

A. * The vectors a and b are orthogonal.

B. The vectors a and b are NOT orthogonal.

C. The cross product of a and b is $a \times b = \begin{pmatrix} 7 \\ -14 \\ -7 \end{pmatrix}$.

D. * The cross product of a and b is $a \times b = \begin{pmatrix} 7 \\ 14 \\ -7 \end{pmatrix}$.

E. * The length of a is $|a| = \sqrt{21}$.

F. The length of a is $|a| = 7$.

8. **(4 points)** A matrix has characteristic polynomial $p(x) = x^2 - 6x + 13$. Find the roots of $p(x)$, i.e., the solutions to $p(x) = 0$.

A. The roots are $x_1 = 3 + \sqrt{22}$ and $x_2 = 3 - \sqrt{22}$.

B. The roots are $x_1 = 3 + \sqrt{2}i$ and $x_2 = 3 - \sqrt{2}i$.

C. There is only one root $x_1 = 3$.

D. The roots are $x_1 = 2 + 3i$ and $x_2 = 2 - 3i$.

E. * The roots are $x_1 = 3 + 2i$ and $x_2 = 3 - 2i$.

F. The roots are $x_1 = -1$ and $x_2 = 7$.

9. **(6 points)** Let A be an $n \times n$ matrix. Which of the following statements are equivalent to “The matrix A is invertible”?

A. * The columns of A are linearly independent.

B. * The determinant of A is nonzero.

C. A has at least one eigenvalue zero.

D. The nullity of A is equal to n.

E. The determinant of A is 0.

F. * The system of linear equations $Ax = b$ has a unique solution for any $b \in \mathbb{R}^n$.

10. **(4 points)** Let
$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},\ B = \begin{pmatrix} 2 & 0 \\ 2 & 2 \end{pmatrix},\ C = \begin{pmatrix} 3 & 3 \\ 0 & 3 \end{pmatrix}.$$
Calculate the matrix product ABC.
A. $\begin{pmatrix} 6 & 12 \\ 1 & 18 \end{pmatrix}$.
B. $\begin{pmatrix} 24 & 24 \\ 2 & 6 \end{pmatrix}$.
C. * $\begin{pmatrix} 12 & 18 \\ 12 & 18 \end{pmatrix}$.
D. $\begin{pmatrix} 6 & 6 \\ 6 & 6 \end{pmatrix}$.
E. $\begin{pmatrix} 6 & 6 \\ 12 & 18 \end{pmatrix}$.
F. $\begin{pmatrix} 3 & 3 \\ 4 & 5 \end{pmatrix}$.

11. **(6 points)** Let A be an $n \times n$ matrix. Which of the following statements are true?
A. * A is diagonalizable if and only if the algebraic multiplicity equals the geometric multiplicity for every eigenvalue.
B. A is diagonalizable if and only if all eigenvalues are distinct.
C. A is always diagonalizable.
D. * A might or might not be diagonalizable.
E. A is diagonalizable if and only if it is Hermitian.
F. A is diagonalizable if and only if $\det A = 0$.

12. **(6 points)** Suppose U is a unitary $n \times n$ matrix. Which of the following is true?
A. * $U^* = U^{-1}$.
B. The determinant of U is real.
C. * All eigenvalues of U have absolute value 1.
D. $U^* = U$.
E. * $|Ux| = |x|$ for all vectors $x \in \mathbb{C}^n$.
F. All eigenvalues of U are real.

13. **(6 points)** Consider the vector space of polynomials of degree smaller or equal 2 with real coefficients. As discussed in class, the set

$$B = \{1, x, x^2\}$$

is a basis of this vector space. Which of the following statements are true?

A. * In the basis B, the polynomial $p(x) = 3x^2 + 5$ has coordinates $(5, 0, 3)$.

B. In the basis B, the polynomial $p(x) = 3x^2 + 5$ has coordinates $(3, 2, 5)$.

C. In the basis B, the linear operator $\frac{d}{dx}$ is represented by the matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

D. * In the basis B, the linear operator $\frac{d}{dx}$ is represented by the matrix $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$.

E. In the basis B, the polynomial $p(x) = 7x + 8$ has coordinates $(7, 8, 0)$.

F. In the basis B, the polynomial $p(x) = 7x + 8$ has coordinates $(7, 1, 8)$.

14. **(4 points)** Consider the matrices

$$S = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{pmatrix}, \quad L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Check whether $S = LU$ is a valid LU decomposition. If the decomposition is valid, then use L and U to compute $\det(S)$.

A. $S \neq LU$.

B. $S = LU$ and $\det(S) = 2$.

C. * $S = LU$ and $\det(S) = 1$.

D. $S = LU$ and $\det(S) = -1$.

E. $S = LU$ and $\det(S) = -2$.

F. $S = LU$ and $\det(S) = 20$.

15. **(6 points)** Which of the following statements is true?

A. Only invertible real square matrices have QR decompositions.

B. * Every real $m \times n$ matrix with $m > n$ has a QR decomposition.

C. * Every invertible real square matrix has a QR decomposition.

D. * Every real square matrix has a QR decomposition.

E. Only real square matrices have QR decompositions.

F. If a matrix has a QR decomposition, then its determinant must be ± 1.

16. **(6 points)** Consider the matrix

$$A = \begin{pmatrix} 1 & 1 & i \\ 1 & 2 & 7 \\ -i & 7 & 4 \end{pmatrix}.$$

Which of the following is true?

A. * A is Hermitian.

B. A is NOT Hermitian.

C. A has purely imaginary eigenvalues.

D. A is unitary.

E. * A is normal.

F. A is NOT normal.

17. **(12 points)**

Compute the inverse of the matrix

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & \frac{1}{2} & 1 \\ 0 & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}.$$

(Here, you need to write down all steps of your solution in order to receive full points.)

Solution: One could use either Gaussian elimination or the classical adjoint to find the inverse. We skip the steps of the solution. The result is

$$A^{-1} = \begin{pmatrix} -3 & 2 & -2 \\ 4 & -2 & 4 \\ 4 & -2 & 2 \end{pmatrix}.$$

18. **(12 points)**

Compute all eigenvalues of the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 6 \\ 0 & 1 & 2 \end{pmatrix}.$$

(Here, you need to write down all steps of your solution in order to receive full points.) Is the matrix A diagonalizable? (Here, you get full point only if you justify your answer correctly.)

Solution: We compute the zeroes of the characteristic polynomial. We find

$$0 = \det \begin{pmatrix} -\lambda & 1 & 0 \\ 0 & 1-\lambda & 6 \\ 0 & 1 & 2-\lambda \end{pmatrix} = (-\lambda)(1-\lambda)(2-\lambda)-(-\lambda)6 = -\lambda^3+3\lambda^2+4\lambda = \lambda(\lambda^2-3\lambda-4).$$

Hence, the eigenvalues are

$$\lambda_1 = -1,\ \lambda_2 = 0,\ \lambda_3 = 4.$$

In class we proved that matrices with all eigenvalues different are diagonalizable. This is the case here, so A is diagonalizable.

19. **(12 points)** Compute a singular value decomposition of the matrix

$$A = \begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix}.$$

(Here, you need to write down all steps of your solution in order to receive full points.)

Solution: We compute

$$AA^T = \begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 8 & 0 \\ 0 & 2 \end{pmatrix}$$

and

$$A^T A = \begin{pmatrix} 2 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix}.$$

The matrix $A^T A$ is already diagonal, and its eigenvalues are $\lambda_1 = 8$, with normalized eigenvector $(1,0)^T$, and $\lambda_2 = 2$, with normalized eigenvector $(0,1)^T$. The singular values are hence $\sqrt{8}$ and $\sqrt{2}$.

The matrix $A^T A$ must have the same eigenvalues as AA^T, i.e., 8 and 2. Corresponding normalized eigenvectors are computed in the following way:

$$\begin{pmatrix} 5-8 & 3 \\ 3 & 5-8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 5-2 & 3 \\ 3 & 5-2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Hence, a possible singular value decomposition is

$$A = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{U} \underbrace{\begin{pmatrix} \sqrt{8} & 0 \\ 0 & \sqrt{2} \end{pmatrix}}_{\Sigma} \underbrace{\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}}_{V^T}.$$